Competitive Programming and Mathematics Society

# Mathematics Workshop

## Methods of Counting

**CPMsoc Maths**

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# Welcome

- We would like to thank everyone for coming, even if its just for the pizza :D
- We are looking forward to expanding our activities from here onwards, if you have any ideas for what you think we can do to satisfy your interests, please let us know!!
- We do have a lot more planned for TERM 2! More Competitive Mathematics heading your way...

# Attendance form

# Our tool kit

- Independent variable acknowledgement, standard Product and Sum rules
  - Given 2 events whose outcomes are independent of one another, we have the the total event outcomes is the product of the number of outcomes of both events.
  - Similarly, the given two complimentary disjoint sets of outcomes to the same event, we take the total outcomes to that event to be the sum of the sets lengths.
- Dependent variable cases
  - For events with a dependence on the outcomes of other events, the total event outcomes can be counted as the sum of the partition of the events into independent groupings
  - e.g.: If $A = \{1, 2, ..., 10\}$, and $B(a) = \{x \in \mathbb{Z}^+ : x < a \in A\}$. The total quantity of groupings of event outcomes $\{a \in A, b \in B\}$ is given by $\sum_{n=1}^{10} |B(n)|$.

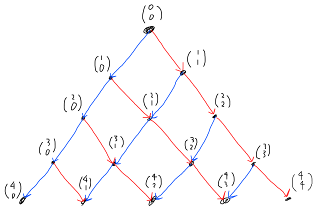
# Pascal's Triangle

- Pascal's triangle is an infinite pattern of numbers where each number is the sum of the two numbers above it. For the following, we define the top row to be the $0^{\text{th}}$ row.
- Some notable properties include:
  - The $n^{\text{th}}$ row has $n+1$ items.
  - The $n^{\text{th}}$ row has a sum of $2^n$;
  - The numbers on the $n^{\text{th}}$ row are $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$ (explained in a later slide).
  - Hence, $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$.
  - From definition, we also see that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.
  - The second outermost diagonal(s) are the counting numbers whilst the third outermost diagonal(s) are the triangular numbers.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

# Pascal's Triangle - The Why

- Consider an ant at the top of Pascal's triangle. It can only descend, but can choose to go either left or right. Each node is marked with how many paths can be taken to reach it.
- Let $n$ to be the length of the path and $k$ be the number of right paths taken.
- Hence, the node which takes $n$ paths to reach and requires $k$ right paths has $\binom{n}{k}$ paths.
- Each outer node has 1 path going towards it and each inner node is also the sum of the two nodes preceding it, satisfying the conditions for Pascal's triangle.



# Repetition, Replacements and Order

For counting repeats of events, there are a couple considerations which change the number of possible outcomes:

- Replacements in counting: The same outcome can be chosen as much as possible
- No replacements: The same outcome cannot be chosen twice
- Ordered Counting: Order of events matter
- Unordered Counting: Outcomes the same up to rearrangement of events

The **choose**/binomial coefficients $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ give the number of ways to select $k$ items from a set of $n$ without replacement and without order.

| Select $k$ from $n$ | Ordered | Unordered |
|---|---|---|
| Without replacement | $\binom{n}{k} \cdot k!$ | $\binom{n}{k}$ |
| With replacement | $n^k$ | $\binom{k+n-1}{k}$ |

Table: Number of ways to select $k$ items from a set of $n$ for cases with and without order and replacement.

# Counting Distributions

The stars-and-bars/sticks-and-balls principle to illuminate ways to distribute objects.

- Finding ways to distribute $n$ balls into $k$ groups is equivalent to finding ways to draw $k-1$ partitions between the possible $n-1$ gaps of the balls: $\binom{n-1}{k-1}$

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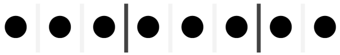
As an example, the number of ways to put $n = 8$ balls in $k = 3$ buckets is equivalent to the number of ways to make $2 = 3 - 1$ partitions in $7 = 8 - 1$ places.



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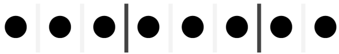

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**Example:** Find the coefficients $a_n$ for the $x^n$ term in the expansion of the following formal series:

$$\left[\sum_{k=1}^{\infty} x^k\right]^m = \sum_{n=m}^{\infty} a_n x^n$$

# Overcounting

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- The *Division rule* helps remove over-counting from equivalent scenarios
  - If every outcome to an event $A$ is counted $n$ times, the total can be divided by $n$ to arrive at the correct counting.

**Example:** Using the division rule and the fact that there are $r!$ ways to arrange $r$ objects, justify why $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

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$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

If we were to select our $k$ objects from our $n$ objects, we would have:

- $n$ ways to select our first object
- $n-1$ ways to select our second
- $n-2$ ways to select our third

and so on. So, the number of ways to select our objects would be:

$$n \times (n-1) \times \cdots \times (n-(k-1)) = \frac{n!}{(n-k)!}.$$

# Overcounting

However, this counts each way to select $k$ objects multiple times! In fact, if we were to select the same $k$ objects in a different order, this method would count it again.

We can see that the number of ways to order our $k$ numbers is $k!$, and so we have to divide our result by $k!$. This gives us:

$$\frac{n!}{k!(n-k)!}$$

# Handshaking Lemma

Here's a famous problem: suppose you're in a room with $20$ people. Each person shakes hands with every other person once and once only. How many handshakes were performed?

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Here's a famous problem: suppose you're in a room with $20$ people. Each person shakes hands with every other person once and once only. How many handshakes were performed?

- Each person shakes hands with $19$ people, so the answer is $20 \cdot 19 = 380$
- This approach does not account for the fact that each handshake will be counted twice.
- We account for this error by dividing $380$ by $2$, so the answer is $190$.

□

# Inclusion-Exclusion

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- The inclusion-exclusion principle is an organized counting technique used to find the size of the union of multiple sets by considering their intersections.
- It states that the size of the union of $n$ sets $A_1, A_2, \ldots, A_n$ can be calculated as follows:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \ldots < i_k \le n} |A_{i_1} \cap \ldots \cap A_{i_k}|$$

- A simple example learnt in high school is $|A \cup B| = |A| + |B| - |A \cap B|$. Inclusion-Exclusion just generalises this concept of adding/subtracting to avoid double-counting for larger numbers of sets.

**Example:** Consider three sets $A$, $B$, and $C$. We want to find the size of their union. Suppose $|A| = 5$, $|B| = 4$, $|C| = 6$, $|A \cap B| = 2$, $|A \cap C| = 3$, $|B \cap C| = 1$, and $|A \cap B \cap C| = 2$.
Applying the inclusion-exclusion principle:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

# Inclusion-Exclusion

**Question:** How many integers between 1 and 1000 are divisible by at least one of 2, 3, or 5?

**Solution:** Let $A_2$, $A_3$, and $A_5$ denote the sets of integers between 1 and 1000 that are divisible by 2, 3, and 5, respectively.
Using the inclusion-exclusion principle, we can calculate the total number of integers divisible by at least one of these numbers as:

$$|A_2 \cup A_3 \cup A_5| = |A_2| + |A_3| + |A_5| - |A_2 \cap A_3| - |A_2 \cap A_5| - |A_3 \cap A_5| + |A_2 \cap A_3 \cap A_5|$$

$$= \left\lfloor \frac{1000}{2} \right\rfloor + \left\lfloor \frac{1000}{3} \right\rfloor + \left\lfloor \frac{1000}{5} \right\rfloor - \left\lfloor \frac{1000}{2 \times 3} \right\rfloor - \left\lfloor \frac{1000}{2 \times 5} \right\rfloor - \left\lfloor \frac{1000}{3 \times 5} \right\rfloor + \left\lfloor \frac{1000}{2 \times 3 \times 5} \right\rfloor$$

$$= 500 + 333 + 200 - 166 - 100 - 66 + 33 = 834$$

Therefore, there are 834 integers between 1 and 1000 that are divisible by at least one of 2, 3, or 5.

# Pigeonhole Principle

- The "classic" Pigeonhole Principle (PHP) states that if we have $n+1$ pigeons and $n$ holes for $n \in \mathbb{N}$ such that every pigeon goes into a hole, *a* hole must have at least $2$ pigeons.
- A generalization (which may also be referred to as pigeonhole principle), is that if we have $km+1$ pigeons and $m$ holes for $k, m \in \mathbb{N}$, then at least one hole must have at least $k+1$ pigeons.
- *Infinite* Pigeonhole Principle states that if there are infinitely many pigeons and finitely many holes, then at least one hole will have infinitely many pigeons.

**Example:** Show that if one selects $n+1$ numbers from the set

$$\{1, 2, \dots 2n\},$$

then there will be some two of them that sum to $2n+1$.

# Pigeonhole Principle

Consider splitting our original set into $n$ subsets which we treat as the "holes":

$$\{1, 2n\}, \{2, 2n-1\}, \ldots, \{n, n+1\}$$

Note that the sum of the elements of each is $2n+1$. By pigeonhole principle, as we select $n+1$ numbers (which act as our pigeons), we will select at least two elements of one of the subsets. This means that we will select some $2$ elements in our original set that sum to $2n+1$, as desired.

- It's crucial to consider what elements will be our "holes", they will be more obscure in harder questions.
- Ensure that each pigeon *is* assigned to a pigeonhole when selecting your "holes".
- When writing up, there is more leniency as long as you state PHP and clearly define what pigeon/holes are (at least with competitions).

# Bijections

- A *bijection* is a mapping $f : \mathcal{A} \longrightarrow \mathcal{B}$ of elements of $\mathcal{A}$ to elements of $\mathcal{B}$ which is both injective and surjective.
- If a bijection exists between some finite sets $\mathcal{A}$ and $\mathcal{B}$, then $|\mathcal{A}| = |\mathcal{B}|$.
- Finding bijections is useful for showing two sets have the same size.

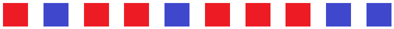
**Example:** Show that

$$\binom{n}{k} = \binom{n}{n-k}$$

for all non-negative integers $n \geq k \geq 0$.

# Bijections

Below is an example of when $n = 10$, $k = 4$ (blue objects are the objects selected).



- Every way for us to choose 4 blue squares has a corresponding way for us to choose the 6 red squares and removing it instead.
- There is a *bijection* between choosing 4 out of 10 blue squares, and 6 out of 10 red squares.
- Thus, the number of ways to choose 4 objects from 10 is the same as choosing 6 objects from 10; i.e, $\binom{10}{6} = \binom{10}{4}$.
- This argument may be generalized for arbitrary non-negative integers $n \geq k \geq 0$.

# Bijections

**Example:**

In Numberland, car plates have six-digit all-number (0-9) plates. If $A$ is the number of cars where the sum of the first three digits is the same as the sum of the last three, and $B$ is the number of cars where all the digits sum to 27, prove that $A = B$.

# Bijections

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**Example:**

In Numberland, car plates have six-digit all-number ($0$-$9$) plates. If $A$ is the number of cars where the sum of the first three digits is the same as the sum of the last three, and $B$ is the number of cars where all the digits sum to $27$, prove that $A = B$.

We can denote each $6$ digit number as $\overline{a_1a_2a_3a_4a_5a_6}$ where $a_1, a_2, \ldots, a_6$ are its digits. We then consider the bijection which maps:

$$\overline{a_1a_2a_3a_4a_5a_6} \mapsto \overline{a_1a_2a_3b_4b_5b_6}$$

where $b_i = 9 - a_i$.

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where $b_i = 9 - a_i$.

Then, we see that if $a_1 + a_2 + a_3 = a_4 + a_5 + a_6$, we get that:

$$\begin{aligned} a_1 + a_2 + a_3 + b_4 + b_5 + b_6 &= a_4 + a_5 + a_6 + b_4 + b_5 + b_6 \\ &= (a_4 + b_4) + (a_5 + b_5) + (a_6 + b_6) \\ &= 27 \end{aligned}$$

and thus if a $6$ digit number has the sum of the first three digits is the same as the sum of the last three, we can biject it to a $6$ digit number whose digit sum is $27$.

# Double Counting

Sometimes to prove an equality, we want to count a quantity in two different ways.

**Example:**
Prove that for all positive integers $n$ and $r$,

$$\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}.$$

We consider the number of ways to pick $r$ people to form a soccer team from $n+1$, and will count this in two ways.

We can just consider the number of ways to pick $r$ people from $n+1$ people to get $\binom{n+1}{r}$ options.

# Double Counting

Alternatively, we can pick a particular person from the $n+1$ people (let's say they're the best player). Then:

- If they aren't chosen to be in the team, there are $\binom{n}{r}$ ways to pick the $r$ people who are.
- If they are chosen to be in the team, there are $\binom{n}{r-1}$ ways to pick the other people $r-1$ players to join the team.

This yields a total of $\binom{n}{r} + \binom{n}{r-1}$ ways to select the soccer team.

Thus we find that

$$\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$$

as they are both the number of ways to select our soccer team.

# Double Counting

**Example:**

Show that

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2$$

for all positive integers $n$.

We consider counting the number of ways to select a team of $n$ people from $2n$ people. By definition, this value will be $\binom{2n}{n}$, which will be our first way of counting it.

# Double Counting

Now, consider splitting our $2n$ people into two groups of $n$ people. For each integer $0 \leq k \leq n$, if we select $k$ people for our team from the first group, we must select $n-k$ people from the other group if we want our team to have $n$ people. Alternatively, we can see this as choosing $k$ people in the other group to not be in our team.

Thus, we find that the number of ways to select our team is

$$\sum_{k=0}^{n} \binom{n}{k} \times \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k}^2$$

and thus the number of ways to choose $n$ objects from $2n$ is also $\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2$, as desired.

# Attendance form :D

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