Invariants and Methods of Counting
2025

UNSW Competitive Programming and Mathematics Society

Counting and Invariants Questions

1. A spider has one sock and one shoe for each of its 8 legs. How many ways can the spider put on all 8 socks and shoes on each leg, assuming that for every leg, each sock is put on before each shoe?
2. Prove that if we have 9 points (no 3 colinear) inside an equilateral triangle of area 1, then we can pick 3 points that form a triangle of area with at most $\frac{1}{4}$.
3. Let n be a positive integer. Prove that the number of partitions of n equals the number of partitions of $2n$ with n parts.
   (A partition of a positive integer n is a way of writing n as the sum of some positive integers. The order of the positive integers does not matter.)
4. The 10-simplex is a 10 dimensional shape with 11 vertices, an edge between every 2 vertices, a triangular face between every 3 vertices and a tetrahedral cell between every 4 vertices. How many edges, faces and cells does the 10-simplex have?
5. Prove that for every positive integer n,
   $$\sum_{i=1}^{n} i\binom{n}{i} = n \times 2^{n-1}$$
6. How many rectangles are there in an 8×8 chessboard?
7. Show that for all positive integers a, b, and n:
   $$\binom{a+b}{n} = \sum_{k=0}^{n} \binom{a}{k}\binom{b}{n-k}$$
8. Prove that for any n integers, we can pick some $m \leq n$ for a positive integer m such that their sum is a multiple of n.
9. A chessmaster has 77 days to prepare for a tournament. He wants to play at least one game per day, but not more than 132 games. prove that there is a sequence of successive days on which he plays exactly 21 games.
10. An n term sequence $(x_1, x_2, \ldots, x_n)$ in which each term is either 0 or 1 is called a binary sequence of length n. Let a_n be the number of binary sequences of length n containing no three consecutive terms equal to $0, 1, 0$ in that order. Let b_n be the number of binary sequences of length n that contain no four consecutive terms equal to $0, 0, 1, 1$ or $1, 1, 0, 0$ in that order. Prove that $b_{n+1} = 2a_n$ for all integers n.