Number Theory Fundamentals 2025

UNSW Competitive Programming and Mathematics Society

Problems

Question 1

Determine the value of $720 \mod 7$.

Question 2

Evaluate gcd(1071, 850).

Question 3

Determine the set of modular residues of $x^{55} \mod 7$ for $x \in \mathbb{Z}$.

Question 4

What is the smallest positive integer x such that the following are all true?:

$$x \equiv 2 \mod 5,$$

$$x \equiv 6 \mod 8,$$

$$x \equiv 4 \mod 22.$$

Question 5

Let $p \leq q$ be coprime natural numbers and suppose you have two jugs with volumes p litres and q litres respectively. Prove that you can measure exactly x litres of water for every natural number $0 \leq x \leq q$.

Question 6

Define the sequence of integers a_i for $i \ge 1 \in \mathbb{Z}$ and $a_1 = 1$ where

$$a_{n+1} = (n+1 - gcd(a_n, n)) \times a_n.$$

Prove that the sequence ratio $\left(\frac{a_{n+1}}{a_n}\right)$ is only equal to n when n is prime or 1. (Simon Marais 2021)

Question 7

Prove that for $m, n \in \mathbb{N}$

 $m^{\varphi(n)} + n^{\varphi(m)} \equiv 1 \pmod{mn}$

wherever the gcd(m, n) = 1.