generating functions problems

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June 2023

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1. Find
$$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

2. How many ways can 30 indistinguishable votes be distributed across 4 different candidates?

3. Find a closed form expression for the nth Fibonacci number using generating functions (hint: have a look at our generating functions workshop slides ;))

4. Let p_n be the nth odd prime number. Show that
$$\prod_{n=1}^{\infty} \left(\frac{p_n^2}{p_n^2 - 1} \right) = \frac{\pi^2}{8}$$

5. The Catalan numbers are generated by the generating function $C(z) = \frac{1-\sqrt{1-4z}}{2z}$. Find a closed formula for the nth Catalan number.

6. Let S be the set of cubic polynomials of the form $x^3 - 17x^2 + ax + b$ which only contain positive integer roots. Find
$$\sum_{p \in S} p(0)$$

7. How many ways can a $3 \times n$ grid be tiled with 2×1 dominoes?

8. [IMO Shortlist 1998] Suppose $a_0, a_1, a_2, \cdots$ is an increasing sequence of non-negative integers such that every non-negative integer can be written as $a_i + 2a_j + 4a_k$, for a unique triple (a_i, a_j, a_k). Find a_{1998}.