Competitive Programming and Mathematics Society

# Mathematics Workshop

## Combinatorics, or Counting with Math

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# Table of contents

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# Welcome

- Next mathematics workshops in week 5.
- Slides will be uploaded on website (unswcpmsoc.com)

# Attendance form

# Counting

- Counting is when you have some things and you want to know how many things you have.
- The naive method is to literally just count them one-by-one, but for large numbers of things this is tedious, so combinatorics can help us count faster.
- A very common application of combinatorics is in probability, where we have mutually exclusive events of equal probability, so that counting the events we care about let us describe the probability of any one of these events occuring as $P(\text{any one of several events occurs}) = \frac{\text{number of events we care about}}{\text{total number of events}}$.

# Counting Example

- In a certain committee, each member belongs to exactly three subcommittees, and each subcommittee has exactly three members. Prove that the number of members equals to the number of subcommittees.

# Counting Example - Solution

- In a certain committee, each member belongs to exactly three subcommittees, and each subcommittee has exactly three members. Prove that the number of members equals to the number of subcommittees.
- Solution. For each of $m$ members, there are 3 member-subcommittee relationships. For each of $n$ subcommittees there are 3 member-subcommittee relationships. So we have $3m = 3n$, and dividing by 3 we are done.

# Factorials

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- We often want to count how many ways some objects can be arranged.
- If we want to arrange $n$ objects into $n$ places, we can place them in one by one.
- The first object can go in $n$ spots. Then there are $n-1$ spots left.
- The second can now go in $n-1$ spots. Then there are $n-2$ spots left.
- If there are $k$ ways of doing one thing, and $q$ ways of doing another independent thing, there are $kq$ ways of doing both,
- Applying this principle in this case, we see there are $n(n-1)(n-2)...3\cdot 2\cdot 1$ ways of arranging these $n$ objects. We call this product $n$ factorial, denoted $n!$.

# Factorials - Example

- A deck of cards is randomly shuffled. What is the chance that after shuffling, the cards are back in the order they started?

# Factorials - Example

- A deck of cards is randomly shuffled. What is the chance that after shuffling, the cards are back in the order they started?
- Answer: They started in one order, but there are $52!$ arrangements they can be shuffled into. Thus, the probability we shuffle them back to the starting order is $\frac{1}{52!}$, which is roughly $1.24 \cdot 10^{-68}$.

# General Rearrangement

- When using factorials, we assumed there were $n$ items in $n$ places and all items were unique. What about when we don't have these assumptions?
- When some $k$ of the items are considered the same, our factorial arrangements consider $k!$ as many cases as they should, because we rearranged these items as if they were different. To counteract this, we can divide out by $k!$ to reverse this extra counting.

# General Rearrangement - Division Example

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- How many strings can be made of the letters "abracadabra"?
- Note that there are 11 total letters, 5 a's, 2 b's, 2 r's, 1 c, and 1 d.

# General Rearrangement - Division Example



- How many strings can be made of the letters "abracadabra"?
- Note that there are 11 total letters, 5 a's, 2 b's, 2 r's, 1 c, and 1 d.
- Answer: we rearrange the 11 letters, but have to divide out our arrangements of the a's, b's, and r's. We get a total of $\frac{11!}{5! \cdot 2! \cdot 2!}$

# Permutation

- What if we have $n$ items, but $k < n$ places? We need to choose which items get to be placed, as some will miss out. Then for the first place, there are $n$ objects to choose from, $n-1$ for the second place, and so on until we have $n-k+1$ objects for the last spot. Multiplying these we get $n(n-1)...(n-k+1)$. This is often denoted $P(n,k) = {}^nP_k$, where P stands for "permutation". Notice this is close to the expression for $n!$, but we have divided out the terms $n-k$, $n-k-1$, and so on. This means we can write ${}^nP_k = \frac{n!}{(n-k)!}$

# Choice

- Notice that when we placed $n$ items in $k < n$ places, we not only chose which items were placed, but we accounted for how they could be arranged in $k!$ ways in these places. If we don't want to count arrangements, and just the number of ways to choose $k$ elements of a set of $n$ objects, we divide ${}^nP_k$ by $k!$ to get the choose function, ${}^nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$.

# Choice - Example

- Prove that for $n \in \mathbb{N}$, $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$. This is called the binomial theorem.

# Choice - Example

- Prove that for $n \in \mathbb{N}$, $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$.
- Answer: $(a+b)^n = (a+b)(a+b)...(a+b)$. When multiplied out, we take from each set of parentheses either an $a$ term or a $b$ term. We do this for $n$ terms, so there are a total of $n$ selections made. If $k$ $a$'s are selected, the other $n-k$ $b$'s are selected. The number of ways of selecting $k$ a's is $\binom{n}{k}$, so this gives the number of $a^k b^{n-k}$ terms.
- Thinking about this example, can we justify why $\binom{n}{k} = \binom{n}{n-k}$

# Inclusion-Exclusion Principle - The Why

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- One fundamental strategy in counting is known as "case bashing". When you are presented with a question which can be broken down into mutually exclusive cases, then you're all good! You only need to add up the individual cases.
- Unfortunately, this is usually not the case. If you are presented with a question with intersecting cases, you must take them into account.
- For example, how many $3$-digit numbers exist such that at least one digit is equal to its position in the number (i.e. the first digit is $1$, the second digit is $2$ and the third digit is $3$)?

# Inclusion-Exclusion Principle - The What

- Let's consider the question posed last slide. There are $100$ numbers with $1$ as its first digit, $90$ numbers with $2$ as its second digit and $90$ numbers with $3$ as its third digit. However, we know that the answer is not $100 + 90 + 90 = 280$.
- We have double counted the numbers $12?$, $1?3$ and $?23$. To fix this, we subtract $280$ by how many numbers are in the form $12?$, $1?3$ and $?23$. There are $10$ numbers in the form $12?$, $10$ numbers in the form $1?3$ and $9$ numbers in the form $?23$. Subtracting them from $280$ gives us $280 - 10 - 10 - 9 = 251$.
- However, we still need to consider $123$. We know we have triple counted it in the first step, and we triple removed it in the second step. Overall, we did not count $123$ in our calculation. Thus, we have to add it back in, hence giving us a total of $251 + 1 = 252$ numbers.

# Inclusion-Exclusion Principle - The How

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- We first define $n$ intersecting sets $A_1, A_2, ...A_n$. Let us denote $f(x)$ as the size of the intersection of $n$ sets.
- For example, $f(2) = |A_1 \cap A_2| + |A_1 \cap A_3| + ... + |A_{n-1} \cap A_n|$.
- Prove that $|A_1 \cup A_2 \cup ... \cup A_n| = \sum_{k=1}^{n} (-1)^{k+1} f(k)$.

# Inclusion-Exclusion Principle - Recipe

- Prove that $|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{k=1}^{n} (-1)^{k+1} f(k)$.
- You can imagine that this formula is a recipe, with the number of steps being equal to the number of sets. The first step is to add in the sizes of the sets, the second step is to subtract by the sizes of the intersections of two sets, and so on.
- We shall prove that for an element $E$ present in exactly $k$ sets, say $E \in A_1 \cap A_2 \cap \ldots \cap A_k$, that $E$ has been $0$-counted if $k$ is odd and double counted if $k$ is even.

# Inclusion-Exclusion Principle - Proof

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- In the first step, $E$ is counted $\binom{k}{1}$ times. In the second step, $E$ is taken away $\binom{k}{2}$ times. Repeating this for the first $k$ steps, we see that $E$ is counted $\binom{k}{1} - \binom{k}{2} + ... + (-1)^{k+1}\binom{k}{k}$ times.
- For any integer $i \geq 1$, we can replace any term $\binom{k}{i}$ with $\binom{k-1}{i-1} + \binom{k-1}{i}$ by virtue of Pascal's triangle.
- Doing so turns our expression into $\binom{k-1}{0} + \binom{k-1}{1} - \binom{k-1}{1} - \binom{k-1}{2} + \binom{k-1}{2} + ...$ and we can see that each term will cancel off. All that matters is finding what the first and last terms are equal to. This technique is known as **telescoping**.

# Inclusion-Exclusion Principle - Proof

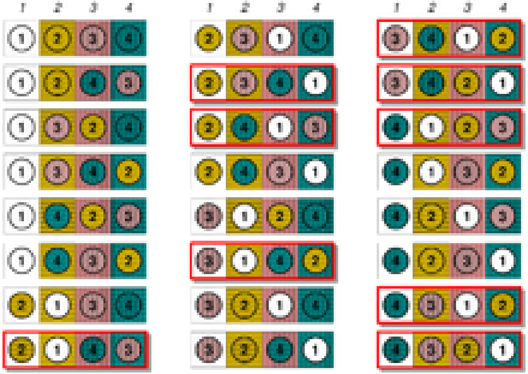
- We can calculate that the first term is $1$ and the last term is $(-1)^k \binom{k}{k} = (-1)^k$. Therefore, it all depends on if $k$ is even or odd.
- If $k$ is odd, it means we have counted $E$ a net total of $1 + (-1)^k = 0$ times and so we adjust this by adding the term $f(k) = (-1)^{k+1} f(k)$.
- If $k$ is even, it means we have counted $E$ a net total of $1 + (-1)^k = 2$ times and so we adjust this by adding the term $-f(k) = (-1)^{k+1} f(k)$.
- In both cases, we have added in the term $(-1)^{k+1} f(k)$. Hence, our final expression will become $\sum_{k=1}^{n} (-1)^{k+1} f(k)$.

# Derangement - Definition

- A derangement (not to be confused with insanity) is a permutation which leaves no elements in its original place.
- For example, DCBA is a derangement of the set ABCD since A is not in the first position, B is not in the second position and so on.
- The notation for the number of derangements of a set of size $n$ is $!n$. Real life uses of derangements include the number of ways to organise a Secret Santa.

# Derangement - Question

- There are many formulae for $!n$, but for the sake of understanding we shall use the formula shown below.
- Prove that for $n \in \mathbb{N}$ and $n > 1$, $!n = n! \sum_{k=0}^{n-2} \frac{(-1)^{n-k}}{(n-k)!}$.
- Note that derangements of size 1 are trivial.



# Derangement - Proof

- Prove that for $n \in \mathbb{N}$ and $n > 1$, $!n = n! \sum_{k=0}^{n-2} \frac{(-1)^{n-k}}{(n-k)!}$.
- This result comes from the inclusion-exclusion principle. Firstly, we define the ordered set $a_1, a_2, ..., a_n$ where $n \in \mathbb{N}$ and $n > 1$.
- Instead of counting the ways where all the elements are in different positions, we count the number of ways where at least one element is fixed.

# Derangement - Proof

- When $a_1$ is fixed, there are $(n-1)!$ ways to arrange the other elements. This similarly applies for $a_2, ..., a_n$ and therefore we get the answer $n \times (n-1)! = n!$.
- However, we double count the cases where we fix two elements $a_1 a_2, a_1 a_3, ..., a_{n-1} a_n$ and so we subtract them. When we fix $a_1 a_2$, there will be $(n-2)!$ ways to arrange the other elements. Repeating this for all other fixing of two elements, we arrive at an answer of $\binom{n}{n-2} \times (n-2)!$.
- Repeating the inclusion-exclusion principle until all elements are fixed will give us the number of ways such that at least one element is fixed, that being $n! - \binom{n}{n-2} \times (n-2)! + \binom{n}{n-3} \times (n-3)! - ... - (-1)^n \times \binom{n}{0} \times 0! = n! - \sum_{k=0}^{n-2} (-1)^{n-k} \binom{n}{k} k!$.
- Subtracting this result from the total number of ways to arrange $n$ elements ($n!$), we get $!n = \sum_{k=0}^{n-2} (-1)^{n-k} \binom{n}{k} k! = n! \sum_{k=0}^{n-2} \frac{(-1)^{n-k}}{(n-k)!}$.

# Pigeonhole Principle

- If I have more pigeons than holes, I can't fit all pigeons in with just one in each hole!
- More generally, with $p$ pigeons and $h$ holes, if $p = qh + r$ for $r > 0$, there must be at least one hole containing $q + 1$ pigeons.
- The main problem solving issue here is identifying what to call a pigeon, and what to call a hole.

# Pigeonhole Principle Example

- Prove that having 100 whole numbers, one can choose 15 of them so that the difference of any two is divisible by 7.

# Pigeonhole Principle Example

- Prove that having 100 whole numbers, one can choose 15 of them so that the difference of any two is divisible by 7.
- Solution: Take the numbers modulo 7 (remainders after division by 7). Then, calling their residues holes and the numbers pigeons, we have $100 = 7 * 14 + 2$, so that some residue must contain 15 members. The difference of any of these 15 equal residues is 0 mod 7, so the difference of the original numbers is divisible by 7.

# Attendance form :D

# Further events

Please join us for:

- Maths workshop in two weeks
- Social session on Friday
- Programming workshop next week