workshop 5 problems

Cyril Subramanian and Haibing Wang

April 2023

1 Introduction

1. Show that $7|x^2 + y^2$ iff $7|x$ and $7|y$ (use quadratic residues).

2. If $7|a^3 + b^3 + c^3$, how many of a, b, c could be divisible by 7? (use cubic residues).

3. Do there exist three squares summing to 7007?

4. Prove there are no integer solutions to

$$x^2 - 2y^2 = 10.$$

5. Find all integer solutions to $a^3 + 2b^3 = 7a^2b$.

6. Prove there are infinite primes 3 mod 4.

7. Given p, q are coprime, find the value of

$$\left\lfloor \frac{p}{q} \right\rfloor + \left\lfloor \frac{2p}{q} \right\rfloor + \ldots + \left\lfloor \frac{(q-1)p}{q} \right\rfloor.$$

8. (Gauss' Lemma) An odd prime p is congruent to 1 mod 4 iff there exists x such that $x^2 \equiv -1$ mod p.

9. Find all consecutive integer powers of 2 and 3 (in either order).

10. For prime p, q, how many quadratic residues are there under mod pq?

11. Prove there are infinite primes 1 mod 4. (a lot harder)