Proof and False Proofs

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1. A knight always tells the truth and a knave always lies. Bob, Jerry and Tom are each either a knight or knave and know who the other knights and knaves are. Bob says everybody is a knight. Jerry says that Bob is a knave, then, shortly afterwards, says Tom is a knave. Out of these three, who is a knight and who is a knave?

2. Find the mistake in this inductive proof that $2^n > n^2$ for all $n \geq 0$.
Base case, $n = 0$: $2^0 = 1 > 0^2 = 0$.
Now we show $2^k > k^2 \implies 2^{k+1} > (k+1)^2$.

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ 2 \cdot 2^k &> 2k^2 \text{ by the inductive step} \\ (k+1)^2 &= k^2 + 2k + 1 \\ k^2 &> 2k + 1 \\ \implies 2k^2 &> k^2 + 2k + 1 = (k+1)^2 \\ \implies 2 \cdot 2^k &> (k+1)^2 \end{aligned}$$

so, inductively we have $2^n > n^2$ for all $n \geq 0$.

3. Given that $a \implies b$ and $b \implies c$ and $d \implies b$, which of the following are necessarily true?
i. $a \implies c$
ii. $a \implies d$
iii. $\neg c \implies a$
iv. $\neg c \implies \neg a$
(Hint: $p \implies q$ statements mean p being true implies q is true, and nothing more. $\neg p$ is read as the negation of p, and is the logical opposite of p, so that if p is true, $\neg p$ is false, and if p is false, $\neg p$ is true.)

4. What is wrong with this proof?

$$\int \frac{1}{x\log(x)}$$
$$u = \frac{1}{\log(x)}, dv = 1/x$$
$$du = \frac{-1}{x\log(x)^2}, v = \log(x)$$
$$\implies \int \frac{1}{x\log(x)} = 1 + \int \frac{1}{x\log(x)}$$
$$\implies 0 = 1$$

5. (SMMC 2022) C1. Let A and B be two fixed positive real numbers. The function f is defined by

$$f(x, y) = min\{x, \tfrac{A}{y}, y + \tfrac{B}{x}\},$$

for all pairs (x, y) of positive real numbers. Determine the largest possible value of $f(x, y)$. (Note: once you have an answer, ensure to try and rigorously prove this solution works).

6. (IMO 2022 P2) Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that there exists exactly one positive real y for every positive real x such that:

$$f(x)y + f(y)x \leq 2$$