Some problemos

CPMSoc

March 2023

1) Is every integer greater than two the sum of two primes?

   Solution: Nope! This may look like the Goldbach conjecture, but note that the actual conjecture only holds for *even* integers greater than 2. For this to hold for an odd integer, it must be the sum of an even prime (which can only be 2) and an odd prime, so we can construct infinite counterexamples by taking an odd composite number and adding 2 (e.g. $3, 11, 17, \ldots$).

2) Isaiah starts with the number 0 and, every turn, randomly adds either 1 or 0. If there number gets to 5, 9 is subtracted from it. What is the expected value of the number they end up with after 9 turns?
   (There is a nice solution that involves no calculations!)

   Solution: For every sequence of 1's and 0's, we can swap each 1 with a 0 and vice versa to obtain a sequence whose final value is exactly the negative of the final value from the first sequence. Since each sequence of 1's and 0's is equally likely, the expected value is 0.

3) (IMC 2022 First Day Problem 1) Let $f : [0, 1] \to (0, \infty)$ be an integrable function such that $f(x)f(1-x) = 1$ for all $x \in [0, 1]$. Prove that

$$\int_0^1 f(x)\mathrm{d}x \geq 1.$$

   Solution: $\int_0^1 f(x)\mathrm{d}x = \int_0^1 f(1-x)dx$ by The King's Property. So:

$$\int_0^1 f(x)\mathrm{d}x = \frac{1}{2}\left(\int_0^1 f(x)\mathrm{d}x + \int_0^1 f(1-x)\mathrm{d}x\right)$$
$$= \frac{1}{2}\left(\int_0^1 (f(x) + f(1-x))\mathrm{d}x\right)$$
$$\geq \frac{1}{2}\left(\int_0^1 2\sqrt{f(x)f(1-x)}\mathrm{d}x\right) = \frac{1}{2}\int_0^1 2\sqrt{1}\mathrm{d}x = 1.$$

   Note that $f(x) > 0$, so the AM-GM inequality holds.

4) A triangle ABC is inscribed in a circle. A line tangent to the circle at point B has a point D chosen on it such that $\angle ABD \geq \angle CBD$. Show that $\angle CBD = \angle CAB$.

Solution: Let O be the center of our circle, and E be the second point of intersection between line OC and the circle. Then, since CE is a diameter, $\angle CBE = 90^\circ$ (angles in a semicircle). Also, $\angle CAB = \angle CEB$ (angles on the same arc BC). $\angle ECB = 180^\circ - \angle EBC - \angle CEB$ (angles in triangle add to 180°), so $\angle ECB = 90^\circ - \angle CAB$. Note that $\angle OBC = \angle ECB$ since they are opposite the equal sides of the icoceles $\triangle OCB$. Finally, since $\angle OBD = 90^\circ$ (angle from tangent to radius of circle), and $\angle OBD = \angle OBC + \angle CBD$, $\angle CBD = 90^\circ - (90^\circ - \angle CAB) = \angle CAB$.

5) Evaluate:

$$\prod_{n=2}^{\infty}\left(1 - \frac{1}{n^2}\right).$$

Solution: Note that

$$\prod_{n=2}^{k}\left(1 - \frac{1}{n^2}\right) = \prod_{n=2}^{k}\frac{(n-1)(n+1)}{n^2}$$

is a telescoping series of product, as each n^2 term will be cancelled out by the $(n-1)$ term after it and the $(n+1)$ term before it. What we will be left with after the cancellation is $\frac{1}{2}\cdot\frac{k+1}{k}$. As $k \to \infty$, $1 + \frac{1}{k} \to 1$, so our answer is just $\frac{1}{2}$.

6) Show there exists a positive Fibonacci number which is divisible by 2023.

Solution: Suppose $F_0 = 0$ and $F_1 = 1$. There are only a finite number of possible ordered pairs of $(x,\ y)$ under $\pmod{2023}$, so eventually, there must exist positive integers $m > n$ such that $F_n \equiv F_m \pmod{2023}$ and $F_{n+1} \equiv F_{m+1} \pmod{2023}$. Note that going from (F_k, F_{k+1}) to (F_{k+1}, F_{k+2}) under $\pmod{2023}$ is a reversible process since if we know F_{k+1} and F_{k+2}, we can *uniquely* deduce $F_k = F_{k+2} - F_{k+1}$. So, $F_{m-n} \equiv F_{n-n} = 0 \pmod{2023}$ and $F_{m+1-n} \equiv F_{n+1-n} = 1 \pmod{2023}$. Thus, since $m + n > 0$, we know there must exist a positive Fibonacci number which is equal to 0 under $\pmod{2023}$.

7) (IMO 2017 P2) Find all functions $f : \mathbb{R} \to \mathbb{R}$ where, for all real x, y:

$$f(f(x)f(y)) + f(x+y) = f(xy).$$

Solution: https://www.cut-the-knot.org/m/Algebra/IMO2017_Problem2.shtml (make sure to look at their second attempt)

8) (Simon Marais 2021 problem B2) Let n be a positive integer. There are n lamps, each with a switch that changes the lamp from on to off, or from

off to on, each time it is pressed. The lamps are initially all off.

You are going to press the switches in a series of rounds. In the first round, you will press exactly 1 switch; in the second, you will press exactly 2 switches; and so on, so that in the kth round you will press exactly k switches. In each round you will press each switch at most once. Your goal is to finish a round with all of the lamps switched on.

Determine for which n you can achieve this goal.

Solution: If we represent turning off a lamp as adding -1 to some sum (which starts at 0) and turning on a lamp as adding 1, if x is the number of times the lamp is turned on and y is the number of times a lamp is turned off, $x+y$ is the total number of lamp flips, and must be a triangular number after each round. $x-y$ is the number of lamps left on at the end, which we require to be n after some number of rounds. These two equations can be solved to find that $x=\frac{n+T_k}{2}$ and $y=\frac{n-T_k}{2}$. If our goal can be achieved for n lamps, then x and y must be non-negative integers. This requires n to have the same parity as T_k and $n \geq T_k$. There is also one hidden assumption: $k \leq n$. This is because we can't have any more rounds after the nth round otherwise, by the pigeonhole principle, at least one lamp will be flipped twice.

An appropriate triangular number can be chosen for $n=1,3,4,5$: respectively, $T_1=1, T_2=3, T_3=6, T_5=15$. $n=2$ cannot have a valid integer x and y chosen since the triangular number T_1 is less than 2 and $T_2=3$ is odd while 2 is even, so our goal cannot be achieved for $n=2$. For $n \geq 5$, we can inductively show that a $T_k \geq n$ can be chosen which is the same parity as n, and also that $k \leq n$. This can be proven inductively since 5 only has an appropriate triangular number after looking at T_3, T_4, T_5, and we will only need to look at a maximum of 3 triangular numbers bigger than or equal to n until we find one satisfying our constraints (this is because even triangular numbers come in groups of two, followed by odd triangular numbers in groups of two, as T_n is even if and only if n or $n+1$ is a multiple of 4). Furthermore, the number of triangular numbers between n and T_n will either stay the same (when n itself is a triangular number) or increase.

Finally, for the positive integers $n \neq 2$, we have shown we can choose appropriate positive integers x, y such that $x+y$ is triangular, $x-y=n$ and no more rounds occur than is necessary. We can now select a sequence of $x+y$ 1's and -1's by adding $1's$ to our list and then adding a -1 when the sum equals n. This way, the sum at any point will always be non-negative but also never exceed n. Then, we can partition the sequence into groups of $1, 2, \cdots$ and appropriately switch a lamp on or off for every number in the sequence. Thus $n \neq 2$ achieve our goal.