Graph Theory (with Linear Algebra) problems

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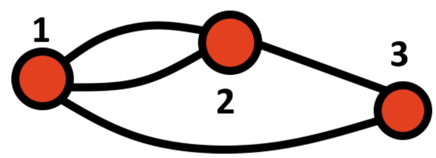
1. Evaluate the matrix product AB where $A = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 4 & 2 \\ 4 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$.
Recall that the ijth entry of the resultant matrix should be the dot product of the ith row of the first matrix with the jth column of the second matrix.

2. How many unique length 4 paths are there between vertices 1 and 2 in the graph below? Confirm this result by raising an adjacency matrix to the 4th power. The 4th power of A can be quickly calculated by calculating A^2 and then squaring the resultant matrix, or just finding A^2 and then taking the dot product of its 1st row and 2nd column.



3. A directed graph is a graph where each edge connects only one way from one vertex to another, typically drawn as an arrow between them. Let the Laplacian matrix L of a simple directed graph G with n vertices be an $n \times n$ matrix where

$$L_{ij} = \left\{ \begin{array}{ll} \text{degree of vertex } v_i & \text{if } i = j \\ -1 & \text{if } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{array} \right\}.$$

Let the incidence matrix A of graph G, under some ordering of the m edges and n vertices, be an $n \times m$ matrix (n rows, m columns) such that

$$A_{ij} = \left\{ \begin{array}{ll} 1 & \text{if edge } e_j \text{ goes into vertex } v_i \\ -1 & \text{if edge } e_j \text{ goes out of vertex } v_i \\ 0 & \text{otherwise} \end{array} \right\}.$$

The transpose operation flips a matrix about the main diagonal (the one going from the top-left to the bottom-right). Formally, the transpose of M, $M^\top$, is the matrix where ${M^\top}_{ij} = M_{ji}$. This operation transforms a $n \times m$ matrix into a $m \times n$ matrix. Show that $AA^\top = L$ for all simple directed graphs G.

4. We call two graphs "isomorphic" if there's a reordering from one graph to the other that preserves their graph "structure". Informally, this occurs if we can move the vertices of one graph around and relabel them without adding, removing or moving edges and get the other graph. How many unique graphs, up to isomorphism, are there with 3 vertices and 4 edges? How about 3 vertices and n edges?

Note: formally, if graphs G and H respectively have vertex sets A, B and edge lists E, F (lists of unordered pairs of vertices, possibly with repeated entries for double edges), then they are isomorphic if there exists a bijective function $f : A \to B$ satisfying $(u, v) \in E \iff (f(u), f(v)) \in F$.

5. We saw an example of a connected graph that had an Euler trail (a trail where all edges are visited exactly once). It had two vertices with odd degree and the rest with even degree. Show that these conditions are sufficient to guarantee the existence of such a trail.

6. Let d_i be the degree of vertex i in an undirected graph. Then, the number walks of length 2 is

$$\sum_{i \in V(G)} d_i^2.$$

a) Prove this with a graph
b) Prove this with an adjacency matrix

7. (Simon Marais 2021) Let $n \geq 2$ be an integer, and let O be the $n \times n$ matrix whose entries are all equal to 0. Two distinct entries of the matrix are chosen uniformly at random, and those two entries are changed from 0 to 1. Call the resulting matrix A.

Determine the probability that $A^2 = O$, as a function of n.

8. Extension: this problem is extension in the sense that it requires content not covered in the workshop, and significant use of external tools possibly unavaliable during competitions. If a graph has adjacency matrix A, and $I - A$ is invertible (I is the appropriately sized identity matrix), can we find a closed form (i.e. with no summation) for the number of length $\leq n$ paths? What other conditions must be met? In particular find a closed form for length $\leq n$ paths between vertices 1 and 4 in the adjacency matrix:

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 3 & 0 & 1 & 0 \end{pmatrix}.$$

Note that to complete this problem fully requires a lot of heavy computation, I recommend using maple if you really want to try this.
You should get $-0.1538461538 - 0.1401205851 * 0.258777175^{n+1} + 0.08250910327 * (-0.258777175)^{n+1} + 0.1330890226 * 3.864328451^{n+1} + 0.07836861309 * (-3.864328451)^{n+1}$.