CPMSoc Elementary Algebra Workshop

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> "Algebra is generous; she often gives more than is asked of her."
>
> — D'Alembert

1 Preliminaries

1.1 Pre-Requisites

1. None

1.2 Notation

1. $\mathbb{N} = \{1, 2, 3 \cdots\}$
2. $\mathbb{P}_n = \{p \in \mathbb{P} : p|n,\ n \in \mathbb{N}\}$
3. Any problem in the problem section that is starred (*) is a standard theorem as well and therefore is highly recommended to be learnt.
4. iff:= if and only if.

2 Algebraic Identities

Familiarity with Algebraic Identities is one of the basic skills that is required when one indulges in Mathematics nevertheless Competitive Mathematics. In this section we present a treatment of some of the most important algebraic identities that one must now.

Lemma 2.1 (Sophie Germain) Let $a, b \in \mathbb{R}$ then

$$a^4 + 4b^4 = (a^2 - 2ab + 2b^2)(a^2 + 2ab + 2b^2).$$

Proof.

$a^4+4b^4+4a^2b^2-4a^2b^2 = (a^2+2b^2)^2-4a^2b^2 = (a^2-2ab+2b^2)(a^2+2ab+2b^2).$

□

Example: Given two line segments of lengths a and b, construct with a straight-edge and compass a segment of length $\sqrt[4]{a^4+b^4}$.

Proof. Given two line segments of length a and b, we have

$$a^4+b^4 = (a^2-\sqrt{2}ab+b^2)(a^2+\sqrt{2}ab+2b^2),$$

hence

$$\sqrt[4]{a^4+b^4} = \sqrt{(a^2-\sqrt{2}ab+b^2)(a^2+\sqrt{2}ab+2b^2)}.$$

Through the law of cosines, we can construct segments of length $\sqrt{a^2 \pm \sqrt{2}ab+b^2}$ using triangles of side a and b with the angle between them being 135 and 45 respectively.
Subsequently we can also construct $\sqrt{xy}$ for "constructible" x and y as this is nothing but the geometric mean given by AD in a right angled triangle ABC (angle(A)=90) with $BD=x$ and $CD=y$. □

Another important Identity that one might be familiar with is

Lemma 2.2 Let $a, b, c \in \mathbb{R}$ then

$$a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca).$$

Proof. Consider the following

$$D = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix},$$

which is to evaluate in two ways first we take the determinant the usual way using Sarrus' rule, and then by adding all the rows and factoring $(a+b+c)$. □

Example: Prove that

$$a^3+b^3+c^3-3abc \geq 0, \ \forall a, b, c \geq 0.$$

Proof. Consider $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$. We take note that $a + b + c \geq 0$ for all $a, b, c \geq 0$. Hence all that remains to be proven is that $(a^2 + b^2 + c^2 - ab - bc - ca) \geq 0$.

Consider through AM-GM,

$$\begin{aligned} a^2 + b^2 &\geq 2ab, \\ b^2 + c^2 &\geq 2bc, \\ c^2 + a^2 &\geq 2ca. \end{aligned}$$

Adding all of the above proves the point. A more direct way is to notice that $(a^2 + b^2 + c^2 - ab - bc - ca) = \frac{1}{2}[(a - b)^2 + (b - c)^2 + (c - a)^2]$which is manifestly non-negative. □

3 Polynomials

The **fundamental theorem of algebra** is a remarkable result about polynomials that was first proven by Carl Gauss in 1799.

Lemma 3.1 [Fundamental Theorem of Algebra] Every polynomial equation of degree n with complex coefficients as at least one complex root.

A corollary of this theorem is that every polynomial with complex coefficients can be written as the product of linear factors with complex in the form $P(z) = A(x - z_1)(x - z_2) \ldots (x - z_n)$ where $z_1, z_2, \ldots, z_n$ are the complex roots (there can be repeated roots, this is referred to as the **multiplicity** of the roots).

Let's have a look at polynomial division now. The following lemma about polynomial division is analogous to the ideas of quotient and remainders we are familiar with when we divide two integers.

Lemma 3.2 [Remainder Theorem] If $A(x)$ and $B(x)$ are polynomials with real coefficients then there exist polynomials $Q(x)$ and $R(x)$ such that

$$A(x) = B(x)Q(x) + R(x)$$

where $\deg(R) < \deg(B)$.

Note: Q and R are respectively called (**quotient**) and (**remainder**).

Now we move onto the factor theorem, which is a simple way of searching for linear factors of polynomials.

Lemma 3.3 [Factor Theorem] If $p \in \mathbb{R}[x]$ then $p(a) = 0$ for some $a \in \mathbb{R}$ if and only if $x - a$ is a factor of $p(x)$.

Proof. If we divide the polynomial $p(x)$ by $x - a$ then we can write $p(x)$ in the form $p(x) = (x - a)q(x) + c$ where $q(x)$ is a polynomial and c is a constant. Note that $p(a) = c$ and we see that $c = 0$ is equivalent to both $p(a) = 0$ and $x - a$ being a factor of $p(x)$. □

Example: Let $A, B, C, D \in \mathbb{R}[x]$ such that

$$A(x^5) + xB(x^5) + x^2C(x^5) = (1 + x + x^2 + x^3 + x^4)D(x), \forall x \in \mathbb{R}.$$

Prove that $(x - 1)$ is a factor of A.

Proof. Consider $x = \omega, \omega^2, \omega^3$, where ω is the fifth root of unity we get

$$\begin{aligned} A(1) + \omega B(1) + \omega^2 C(1) = 0 \\ A(1) + \omega^2 B(1) + \omega^4 C(1) = 0 \\ A(1) + \omega^3 B(1) + \omega^6 C(1) = 0 \end{aligned}$$

therefore we have that $A(1) = B(1) = C(1) = 0$ by solving the simultaneous equations and therefore using the factor theorem we have that $x - 1$ is a factor of $A(x)$. □

A common lemma that is quite elementary to the study of polynomials is the Gauss' Lemma which gives a necessary and sufficient condition for the irreducibility of integer polynomials.

4 Inequalities

The triangle inequality is a simple but very useful inequality involving absolute values.

Lemma 4.1 (Triangle Inequality) Let $x, y \in \mathbb{R}$. Then

$$|x + y| \leq |x| + |y|, \ \forall x, y \in \mathbb{R}.$$

Proof. Squaring both sides means that the inequality is equivalent to

$$x^2 + y^2 + 2xy \leq x^2 + y^2 + 2|xy|,$$

which is true because $|xy| \geq xy$.

Corollary: We can prove a generalization too using induction, that

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|, \ \forall x_i \in \mathbb{R}.$$

Corollary: (Triangle Inequality [Complete]) Let $x, y \in \mathbb{R}$. Then

$$||x| - |y|| \leq |x + y| \leq |x| + |y|, \ \forall x, y \in \mathbb{R}.$$

□

We also note that this is true for normed spaces and in general the p-norm, that for all $x, y \in \mathbb{R}^n$ we have

$$||x + y||_p \leq ||x||_p + ||y||_p,$$

where $||x||_p = (\sum_{i=1}^n x_i^p)^{1/p}$.

Another useful inequality is the Arithmetic-Geometric mean (AM-GM) inequality.
One of the other inequalities that is quite often usefull in mathematics is the Arithmetic-Geometric Inequality or commonly known as AM-GM.

Lemma 4.2 (AM-GM Inequality) Let $x_1, \cdots, x_n$ be positive reals. Then

$$\frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n}.$$

Proof. We begin by using the fact that

$$\log(x) \leq x - 1, \forall x > 0.$$

Which follows due to the MVT (Try this). Let $a_1, \cdots, a_n$ be positive reals and define A to be the arithematic mean i.e.

$$A = \frac{a_1 + \cdots + a_n}{n}.$$

Consider $x = \frac{a_i}{A}$, therefore

$$\log\left(\frac{a_i}{A}\right) \leq \frac{a_i}{A} - 1 \implies \log\left(\frac{a_1 a_2 \ldots a_n}{A^n}\right) \leq 0.$$

□

A useful inequality that involves sequences of numbers is the **rearrangement inequality**, which aims to maximise or minimise the sum of products of corresponding terms in two sequences.

Lemma 4.3 (Rearrangement inequality) Let $(a_1, a_2, \cdots, a_n)$ and $(x_1, x_2, \cdots, x_n)$ be two sequences of real numbers. Then the permutation $(b_1, b_2, \cdots, b_n)$ of $(x_1, x_2, \cdots, x_n)$ which maximises the expression

$$E = a_1b_1 + a_2b_2 + \ldots + a_nb_n$$

is the permutation where the sequences $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are sorted the same way. The permutation that minimises the expressions is where the sequences are sorted the opposite way.

The **Cauchy-Schwarz** inequality is another powerful inequality that involves two sequences of real numbers and has many generalisations.

Lemma 4.4 (Cauchy-Schwarz inequality) If $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are two sequences of real numbers, then

$$(x_1^2 + x_2^2 + \ldots + x_n^2)(y_1^2 + y_2^2 + \ldots + y_n^2) \geq (x_1y_1 + x_2y_2 + \ldots + x_ny_n)^2$$

Equality holds if and only if we have the equal ratios

$$\frac{x_1}{y_1} = \frac{x_2}{y_2} = \cdots = \frac{x_n}{y_n}$$

5 Weighted Inequalities

There are more general versions of many classical inequalities that involves weights.

Lemma 5.1 (Weighted AM-GM Inequality) For positive real numbers $a_1, a_2, \ldots, a_n$ and positive real numbers $w_1, w_2, \ldots, w_n$ (called weights), we have the inequality

$$\frac{w_1a_1 + w_2a_2 + \ldots + w_na_n}{w_1 + w_2 + \ldots + w_n} \geq \sqrt[w_1+w_2+\ldots+w_n]{a_1^{w_1}a_2^{w_2}\ldots a_n^{w_n}}$$

with equality if and only if $a_1 = a_2 = \ldots = a_n$.

6 Problems

6.1 Introductory Problems

1. Let a and b be coprime integers greater than 1. Prove for any $n \geq 0$, $a^{2n+1} + b^{2n+1}$ is divisible by $a + b$.

2. Find the remainder when dividing the polynomial $x^{100} - 2x^{51} + 1$ by $x^2 - 1$.

3. Let $f(x) \in \mathbb{R}[x]$, and suppose that $f(x) + f'(x) > 0$ for all x. Prove that $f(x) > 0$ for all x.

4. Evaluate the determinant

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ w & x & y & z \\ w^2 & x^2 & y^2 & z^2 \\ w^3 & x^3 & y^3 & z^3 \end{vmatrix}.$$

6.2 Intermediate Problems

1. Prove that for all positive integers n, the polynomial $1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}$ has no multiple roots.

2. What is the minimum possible value of $x+y+z$ where x, y, z are positive real numbers satisfying $xy^2z^3 = 108$?

3. Prove that for $x, y \in \mathbb{R}^n$,

$$||x + y||_p \leq ||x||_p + ||y||_p,$$

where $||x||_p$ is the p-norm.

4. Let $P(x)$ be a real polynomial such that $P(x) \geq 0$ for all real x. Prove that it is possible to write

$$P(x) = F(x)^2 + G(x)^2$$

for all real polynomials $F(x)$ and $G(x)$.

6.3 Advanced Problems

1. Show that the solution set of the inequality

$$\sum_{k=1}^{70} \frac{k}{x-k} \geq \frac{5}{4}$$

is a union of disjoint intervals, the sum of whose length is 1988.

2. Find all pairs of positive integers $m, n \geq 3$ for which there exist infinitely many positive integers a such that

$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$

is itself an integer.

3. Prove that the inequality

$$\left(a^2 + 2\right)\left(b^2 + 2\right)\left(c^2 + 2\right) \geq 9\left(ab + bc + ca\right)$$

holds for all positive reals a, b, c.