Competitive Programming and Mathematics Society

Combinatorial Geometry

Workshop 3, Week 7, Term 3, 2021

CPMSoc Mathematics

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Combinatorial Geometry

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Combinatorial geometry is a hybrid between the topics of combinatorics and geometry.

Often, combinatorial geometry problems are posed in geometric terms and but require a combination of geometric and combinatorial ideas in their solution.

While classical geometry problems tend to have only a small number of points in the diagram, combinatorial geometry problems can often have any arbitrary number of points and you may be asked to prove a statement for large numbers of points.

Using Proof by Contradiction

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Example

Is it possible to colour every point on a circle using the two colours white and black so that there is no isosceles triangle whose vertices all have the same colour?

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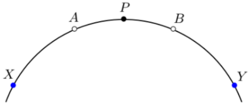

So let's assume that it is possible to aim for a contradiction. Then we can choose two points A and B of the same colour, say white.

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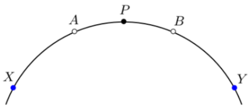

So let's assume that it is possible to aim for a contradiction. Then we can choose two points A and B of the same colour, say white.

Choose point X so that A is the midpoint of arc $\overset{\frown}{XB}$ to create the isosceles triangle $\triangle AXB$, which forces X to be blue.

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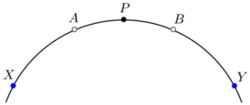

Similarly, if we choose point Y to satisfy $AB = BY$ and B between A and Y, then Y must also be blue.

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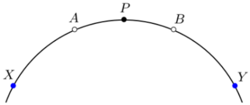

Similarly, if we choose point Y to satisfy $AB = BY$ and B between A and Y, then Y must also be blue.

Choose P to be on the minor arc of the circle halfway between A and B. Then triangles $\triangle ABP$ and $\triangle XPY$ are isosceles so we cannot colour P anything, which is a contradiction.

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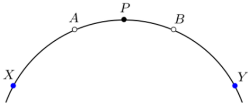

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There is a slight gap in the proof in the cases where the five points are not all distinct. These cases can be solved separately.

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Another solution is to consider five points equally spaced around the circle and noticing that any three of the points form an isosceles triangle.

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Another solution is to consider five points equally spaced around the circle and noticing that any three of the points form an isosceles triangle.

Applying the pigeonhole principle leads to the conclusion.

Extremal Principle

Sometimes we can pick a very specific object in a combinatorial geometry problem that is "extremal" in some sense, such as being the greatest / least or the best / worst with respect to some property.

Extremal Principle

Example

We are given a set of discs in the plane with pairwise disjoint interiors. Each disc is tangent to at least six other discs in the family. Prove that there are infinitely many discs in the set.

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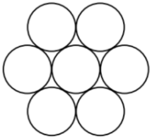

But there is only room for there to be exactly six discs, all of radius r around D.

Extremal Principle

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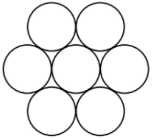
If we apply the same argument to each of these discs and so on, we can expand outwards forever to generate infinitely many discs of radius r, giving us a contradiction.



Perturbation

Sometimes objects such as points or lines may not be exactly in the position that we want them in. Often it is possible to shuffle the configuration slightly to rectify this.

Perturbation

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Given n points in the plane, no three of which are collinear, show it is possible to join them up in sequence so that we have a broken line consisting of $n - 1$ segments, no two of which cross each other.

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The points lie in the x, y plane. If we label the points $P_1, P_2, \ldots, P_n$ according to increasing x-coordinate, then we could simply join P_1 to P_2, P_2 to P_3, and so on.

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However, we cannot guarantee that the x-coordinates are all distinct. Perhaps we could rotate the configuration in the plane so that all the x-coordinates are distinct?

Indeed, all we have to do is to rotate the configuration so that the y-axis is not parallel to any of the lines formed by joining all $\binom{n}{2}$ pairs of points, so this completes the problem.

Induction

Often, combinatorial geometry problems where we can build examples with a larger set of points from a smaller set of points can be approached by mathematical induction.

Induction

Example

Given 1002 distinct points in the plane, we join every pair of points with a line segment and colour its midpoint red.

Show that there are at least 2001 red points.

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We prove by induction that for $n \geq 2$ points there are at least $2n - 3$ red points. This is clearly true for $n = 2$.

Suppose that the result is true for $n = 2, 3, \ldots, m$ where $m \geq 2$. Consider an arrangement of $m + 1$ points which we can assume to have distinct $x-$coordinates by a perturbation argument. Label these points as $A_1, A_2, \ldots A_m, A_{m+1}$ by increasing $x-$coordinate.

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By assumption we have at least $2m-3$ red points from the midpoints of $A_1, A_2, \ldots, A_m$.

The midpoints of $A_{m+1}A_{m-1}$ and $A_{m+1}A_m$ are distinct and both are to the right of all the red points considered so far. This gives us at least $2m-1 = 2(m+1)-3$ red points in all, which completes the proof.

Convex Hull

Consider a wooden board with some nails hammered into it. When you take a rubber band and stretch it around the area occupied by the nails, you end up with a polygon whose angles are all less than or equal to $180°$.

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The general notion of a convex hull is defined as follows. A set S is convex if for any two points A and B in S, the whole line segment AB lies entirely in S. It can be proven that the intersection of convex sets is also a convex set.

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The *convex hull* of a set of points T is defined as the intersection of all convex sets containing T. It is also the smallest convex set containing T.

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Example

Let S be a set of five distinct points in the plane. Show that there exist three points A, B, C such that $108^\circ \leq \angle ABC \leq 180^\circ$.

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Consider the perimeter of the convex hull. If it is five points (forming a convex pentagon) then since the angle sum of the pentagon is 540° then at least one of the interior angles must be between $540^\circ/5 = 108^\circ$ and 180°.

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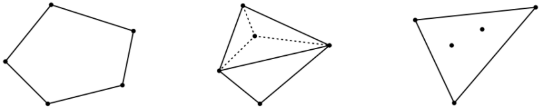

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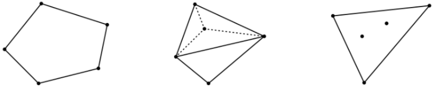

If the convex hull is a quadrilateral then divide it into two triangles as shown.

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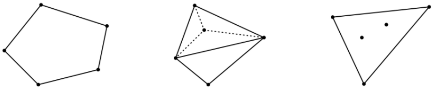

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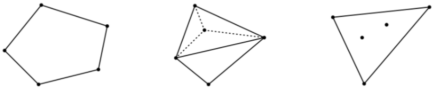

If the convex hull is a quadrilateral then divide it into two triangles as shown.
The fifth point must be inside one of the triangles. Join the fifth point to the vertices of the triangle it lies inside.
Now consider the three angles around the fifth point created by the three segments.
These angles are each at most 180° and add up to 360° so we can find an angle that is also at most $360^\circ/3 = 120^\circ$.

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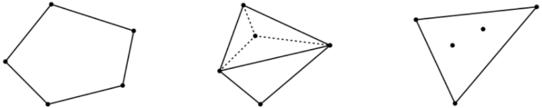

If the convex hull is a triangle then again we have a triangle with a point inside it (in fact two points) in its interior, so we can find an angle in the desired range using a similar argument as before.

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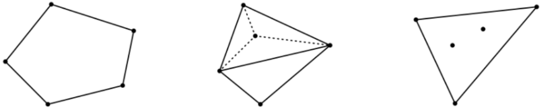

If the convex hull is a triangle then again we have a triangle with a point inside it (in fact two points) in its interior, so we can find an angle in the desired range using a similar argument as before.

If the convex hull is a line segment, then all the points are collinear so any three of them form a straight angle.

Pigeonhole Principle

Example

Six points are given inside an equilateral triangle of area 4. Prove that among nine points which include the three vertices of the triangle and the six points, three of these form a triangle of area at most 1.

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Divide the triangle up into four equilateral triangles of area 1.

Since there are nine points altogether then by the pigeonhole principle at least three of these points lie inside or on the boundary of one of these four triangles and thus define a triangle of area at most 1.

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In fact more is true! We can sharpen the result from 1 to $\frac{4}{13}$ as follows.

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Place the first point inside the triangle and use this point to subdivide the original triangle into three smaller triangles.

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Place the first point inside the triangle and use this point to subdivide the original triangle into three smaller triangles.

Next place the second point. This will fall inside one of the three smaller triangles, or on one (inner) edge. Either way, we will increase the number of triangles by 2 each time.

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Next place the second point. This will fall inside one of the three smaller triangles, or on one (inner) edge. Either way, we will increase the number of triangles by 2 each time.

Continuing this procedure leads us to subdivide the original triangle into 13 smaller triangles using the six points. Thus, one of the triangles will have area at most $\frac{4}{13}$.