Competitive Programming and Mathematics Society

Functional Equations

Term 3, Workshop 1

CPMSoc Mathematics

What is a Functional Equation

Functional Equations

Functional equations are equations where the unknowns are functions, rather than a traditional variable.

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For example:

1. $f(xy) = f(x)f(y)$
2. $f(x)f(y) = f(x+y)$
3. $f(x) + f(y) = f(xy)$
4. $f(x+y) = f(x) + f(y)$
5. $f(x+y) = \frac{f(x)+f(y)}{1-f(x)f(y)}$.

Functional Equations

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4. $f(x+y) = f(x) + f(y)$
5. $f(x+y) = \frac{f(x)+f(y)}{1-f(x)f(y)}$.

Can you guess a possible solution to these functional equations?

Functional Equations

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Example

Find all functions f such that $f : \mathbb{Q} \longrightarrow \mathbb{Q}, f(1) = 2, f(xy) = f(x)f(y) - f(x+y) + 1.$

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2. the value of f at some number(s)

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Three of the most common things mentioned in functional equation problems:

1. the domain and codomain
2. the value of f at some number(s)
3. the main functional equation(s).

Cauchy's Functional Equation

Cauchy's functional equation asks us to find functions which satisfy

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Substituting $x = y = 0$ yields

$$f(0) + f(0) = f(0)$$

which then implies that

$$f(0) = 0.$$

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We can try guessing the value of $f(1)$ but with $f(x) = cx$ as a possible solution, we let $f(1) = c$ and try obtaining $f(2)$,

$$f(2) = f(1) + f(1) = 2c.$$

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$$f(2) = f(1) + f(1) = 2c.$$

In fact, putting $y = 1$ in the functional equation gives

$$f(x+1) = f(x) + f(1) = f(x) + c.$$

From here you should be able to prove by induction that $f(x) = cx$ for all non-negative integers x and some real number c.

Cauchy's Functional Equation

Since we have already deduced that $f(0) = 0$, it makes sense to try the substitution $y = -x$ in the functional equation. This leads to

$$f(0) = f(x) + f(-x) \implies f(-x) = -f(x).$$

This piece of information tells us that $f(x) = cx$ holds for every integer, whether positive, negative or zero.

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Now for any integer m and positive integer n, we have

$$f\left(\frac{m}{n} + \frac{m}{n} + \cdots + \frac{m}{n}\right) = f\left(\frac{m}{n}\right) + f\left(\frac{m}{n}\right) + \cdots + f\left(\frac{m}{n}\right)$$

Cauchy's Functional Equation

Since we have already deduced that $f(0)=0$, it makes sense to try the substitution $y=-x$ in the functional equation. This leads to

$$f(0)=f(x)+f(-x) \implies f(-x)=-f(x).$$

This piece of information tells us that $f(x)=cx$ holds for every integer, whether positive, negative or zero.

Now for any integer m and positive integer n, we have

$$f\left(\frac{m}{n}+\frac{m}{n}+\cdots+\frac{m}{n}\right)=f\left(\frac{m}{n}\right)+f\left(\frac{m}{n}\right)+\cdots+f\left(\frac{m}{n}\right)$$

Therefore $f(m)=nf\left(\frac{m}{n}\right)$ which implies that

$$f\left(\frac{m}{n}\right)=\frac{f(m)}{n}=\frac{cm}{n}.$$

We've now deduced that $f(x)=cx$ for all rational numbers x and some real number c.

Guess and Hope

Example. Find all functions $f : \mathbb{Q} \to \mathbb{R}$ such that

$$f(x + y) = f(x) + 2xy + f(y)$$

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$$(x+y)^2 = x^2 + 2xy + y^2 \text{ (!)}$$

This verifies that $f(x) = x^2$ is a solution but is it the only solution?

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$$(x+y)^2 = x^2 + 2xy + y^2 \text{ (!)}$$

This verifies that $f(x) = x^2$ is a solution but is it the only solution? Now let's substitute $f(x) = g(x) + x^2$ in the hope that $g(x)$ satisfies a simpler functional equation.

Guess and Hope

This leads to

$$g(x+y)+(x+y)^2=g(x)+g(y)+x^2+2xy+y^2$$

which leaves

$$g(x+y)=g(x)+g(y)$$

Guess and Hope

This leads to

$$g(x+y) + (x+y)^2 = g(x) + g(y) + x^2 + 2xy + y^2$$

which leaves

$$g(x+y) = g(x) + g(y)$$

This is Cauchy's functional equation! The solutions to this are given by $g(x) = cx$ and so

$$f(x) = x^2 + cx.$$

Substitutions

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Example. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \neq 0$ for $x \neq 0$ and

$$f(f(x) + y) = f(x^2 - y) + 4f(x)y$$

for all real numbers x and y.

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We would like to find algebraic substitutions which provide nice cancellation. For example, the $f(f(x) + y)$ motivates us to try $y = -f(x)$, which yields

$$f(0) = f(x^2 + f(x)) - 4f(x)^2.$$

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We would like to find algebraic substitutions which provide nice cancellation. For example, the $f(f(x) + y)$ motivates us to try $y = -f(x)$, which yields

$$f(0) = f(x^2 + f(x)) - 4f(x)^2.$$

Furthermore, the term $f(x^2 - y)$ motivates us to try $y = x^2$, which yields

$$f(f(x) + x^2) = f(0) + 4f(x)x^2.$$

Substitutions

Now, both of the equations,

$$f(0) = f(x^2 + f(x)) - 4f(x)^2.$$

$$f(f(x) + x^2) = f(0) + 4f(x)x^2.$$

have the term $f(f(x) + x^2)$

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have the term $f(f(x) + x^2)$ and by eliminating this, we get

$$f(0) + 4f(x)^2 = f(0) + 4f(x)x^2 \Rightarrow f(x)^2 = f(x)x^2.$$

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$$f(x)(f(x) - x^2) = 0.$$

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$$f(x)(f(x) - x^2) = 0.$$

Hence the solution is $f(x) = x^2$ as $f(x) \neq 0$ for $x \neq 0$

Injective, Surjective and Bijective

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A function is said to be bijective if it is both injective and surjective. The big advantage of having a bijective function f is that there exists an inverse function f^{-1} which satisfies

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(x)) = x$$

.

Injective, Surjective and Bijective

Example. The function $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$f(xf(x) + f(y)) = f(x)^2 + y$$

for all real numbers x and y. Prove that f is bijective.

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Surjectivity. We let $x = 0$ and vary y. This gives

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Surjectivity. We let $x = 0$ and vary y. This gives

$$f(f(y)) = f(0)^2 + y.$$

By substituting $b = f(0)^2 + y$, we get

$$f(f(b - f(0)^2)) = b$$

and so

$$f(a) = b$$

where $a = f(b - f(0)^2)$.

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$$f(f(b - f(0)^2)) = b$$

and so

$$f(a) = b$$

where $a = f(b - f(0)^2)$. Since there is an a for every b such that $f(a) = b$, $f(x)$ is surjective.

Injective, Surjective and Bijective

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Injectivity. We assume that $f(y_1) = f(y_2)$ and hope to deduce that $y_1 = y_2$. But if $f(y_1) = f(y_2)$, then we have

Injective, Surjective and Bijective

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Injectivity. We assume that $f(y_1) = f(y_2)$ and hope to deduce that $y_1 = y_2$. But if $f(y_1) = f(y_2)$, then we have

$$f(xf(x) + f(y_1)) = f(xf(x) + f(y_2))$$

$$\Rightarrow f(x)^2 + y_1 = f(x)^2 + y_2.$$

Injective, Surjective and Bijective

Injectivity. We assume that $f(y_1) = f(y_2)$ and hope to deduce that $y_1 = y_2$. But if $f(y_1) = f(y_2)$, then we have

$$f(xf(x) + f(y_1)) = f(xf(x) + f(y_2))$$

$$\Rightarrow f(x)^2 + y_1 = f(x)^2 + y_2.$$

Therefore, we can conclude that $y_1 = y_2$ and so f must be injective.

Since we have shown that f is both injective and surjective, we now know that f is bijective.

The associative trick

It is the idea that $f(g(h(x)))$ can be evaluated in two different ways

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Example. Do there exist functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that

$$f(g(x)) = x^2 \quad \text{and} \quad g(f(x)) = x^3$$

for all real numbers x?

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for all real numbers x?

The following chain of implications tells us that the function f must be injective.

$$f(a) = f(b) \Rightarrow g(f(a)) = g(f(b)) \Rightarrow a^3 = b^3 \Rightarrow a = b.$$

The Associative trick

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Therefore,

$$f(x)^2 = f(x^3)$$

for all values of x. In particular, we know that

$$f(-1) = f(-1)^2, \quad f(0) = f(0)^2 \quad \text{and} \quad f(1) = f(1)^2$$

What does this mean?

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Contradiction!!! f is injective.

Exploit Symmetry

Example. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f(m + f(n) + mf(n)) = m + mn + f(n)$$

for all integers m and n.

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Substituting $m = f(p)$ for any integer p,

$$f(f(p) + f(n) + f(p)f(n)) = f(p) + f(p)n + f(n)$$

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$$f(f(p) + f(n) + f(p)f(n)) = f(p) + f(p)n + f(n)$$

What do you notice? LHS Symmetry

This means RHS Symmetry!

$$f(p) + f(p)n + f(n) = f(n) + f(n)p + f(p)$$

Exploit Symmetry

Hence

$$f(p)n = f(n)p$$

for all integers p and n.

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for all integers p and n.

Substituting $p = 1$, we deduce that $f(n) = f(1)n = cn$.

Exploit Symmetry

Hence

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for all integers p and n.

Substituting $p = 1$, we deduce that $f(n) = f(1)n = cn$.

We must check our solutions by plugging $f(n) = cn$ into the original functional equation. We deduce that $c = 1$. So, the only possible solution is $f(n) = n$.

Involutions

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Example. Find all functions $g : \mathbb{R} \setminus \{\frac{2}{3}\} \to \mathbb{R}$ such that

$$x - g(x) = \frac{1}{2} g\left(\frac{2x}{3x - 2}\right)$$

for all real numbers $x \neq \frac{2}{3}$

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This functional equation compares two different values at g at one at x and one at $\frac{2x}{3x-2}$.

If x is related to $\frac{2x}{3x-2}$, then what is $\frac{2x}{3x-2}$ related to?

Involutions

Let's try replacing x with $\frac{2x}{3x-2}$:

$$\frac{2x}{3x-2} - g\left(\frac{2x}{3x-2}\right) = \frac{1}{2}g\left(\frac{2(\frac{2x}{3x-2})}{3(\frac{2x}{3x-2})-2}\right)$$

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A little algebra reveals that

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A little algebra reveals that

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So we now have the equations

$$x - g(x) = \frac{1}{2}g\left(\frac{2x}{3x-2}\right) \quad \text{and} \quad \frac{2(\frac{2x}{3x-2})}{3(\frac{2x}{3x-2})-2} = \frac{4x}{6x-2(3x-2)} = x.$$

Involutions

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You can think of these as two simultaneous equations, which you can solve for both $g(x)$ and $g\left(\frac{2x}{3x-2}\right)$.

Involutions

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Whats going on here?

References I

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Thanks for listening!