Competitive
Programming and
Mathematics
Society

Linear Algebra

Workshop 3, Week 7, Term 2, 2021

CPMSoc Mathematics

Table of contents

Linear Mappings

CPMSOC

A mapping $T : V \to W$ between vector spaces that share the same field is called *linear* when:

- $T(\overrightarrow{u} + \overrightarrow{v}) = T(u) + T(v)$
- $T(a\overrightarrow{v}) = aT(\overrightarrow{v})$

Linear Mappings

A mapping $T : V \to W$ between vector spaces that share the same field is called *linear* when:

- $T(\overrightarrow{u} + \overrightarrow{v}) = T(u) + T(v)$
- $T(a\overrightarrow{v}) = aT(\overrightarrow{v})$

Alternatively, $T(a\overrightarrow{u} + b\overrightarrow{v}) = aT(\overrightarrow{u}) + bT(\overrightarrow{v})$.

Linear Mappings

A mapping $T : V \to W$ between vector spaces that share the same field is called *linear* when:

- $T(\overrightarrow{u} + \overrightarrow{v}) = T(u) + T(v)$
- $T(a\overrightarrow{v}) = aT(\overrightarrow{v})$

Alternatively, $T(a\overrightarrow{u} + b\overrightarrow{v}) = aT(\overrightarrow{u}) + bT(\overrightarrow{v})$.

Example

Show that the derivative is a linear mapping from the set of infinitely differentiable functions, $\mathcal{C}^\infty$, to itself.

Linear Mappings

A mapping $T : V \to W$ between vector spaces that share the same field is called *linear* when:

- $T(\overrightarrow{u} + \overrightarrow{v}) = T(u) + T(v)$
- $T(a\overrightarrow{v}) = aT(\overrightarrow{v})$

Alternatively, $T(a\overrightarrow{u} + b\overrightarrow{v}) = aT(\overrightarrow{u}) + bT(\overrightarrow{v})$.

Example

Show that the derivative is a linear mapping from the set of infinitely differentiable functions, $\mathcal{C}^\infty$, to itself.

$D(af(x) + bg(x))$

Linear Mappings

A mapping $T : V \to W$ between vector spaces that share the same field is called *linear* when:

- $T(\overrightarrow{u} + \overrightarrow{v}) = T(u) + T(v)$
- $T(a\overrightarrow{v}) = aT(\overrightarrow{v})$

Alternatively, $T(a\overrightarrow{u} + b\overrightarrow{v}) = aT(\overrightarrow{u}) + bT(\overrightarrow{v})$.

Example

Show that the derivative is a linear mapping from the set of infinitely differentiable functions, $\mathcal{C}^\infty$, to itself.

$D(af(x) + bg(x)) = \lim_{h\to 0} \frac{(af(x+h)+bg(x+h))-(af(x)+bg(x))}{h}$

Linear Mappings

A mapping $T : V \to W$ between vector spaces that share the same field is called *linear* when:

- $T(\overrightarrow{u} + \overrightarrow{v}) = T(u) + T(v)$
- $T(a\overrightarrow{v}) = aT(\overrightarrow{v})$

Alternatively, $T(a\overrightarrow{u} + b\overrightarrow{v}) = aT(\overrightarrow{u}) + bT(\overrightarrow{v})$.

Example

Show that the derivative is a linear mapping from the set of infinitely differentiable functions, $\mathcal{C}^{\infty}$, to itself.

$D(af(x) + bg(x)) = \lim_{h\to 0} \frac{(af(x+h)+bg(x+h))-(af(x)+bg(x))}{h} =$
$a\left(\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\right) + b\left(\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}\right)$

Linear Mappings

A mapping $T : V \to W$ between vector spaces that share the same field is called *linear* when:

- $T(\overrightarrow{u} + \overrightarrow{v}) = T(u) + T(v)$
- $T(a\overrightarrow{v}) = aT(\overrightarrow{v})$

Alternatively, $T(a\overrightarrow{u} + b\overrightarrow{v}) = aT(\overrightarrow{u}) + bT(\overrightarrow{v})$.

Example

Show that the derivative is a linear mapping from the set of infinitely differentiable functions, $\mathcal{C}^{\infty}$, to itself.

$D(af(x) + bg(x)) = \lim_{h\to 0} \frac{(af(x+h)+bg(x+h))-(af(x)+bg(x))}{h} =$
$a\left(\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\right) + b\left(\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}\right) = aD(f(x)) + bD(g(x))$.

Matrix Representation Theorem

CPMSOC

CLUBS SUPPORTED BY Arc INDEPENDENTLY RUN

If a vector space has a basis $e_1, e_2, \ldots$, we can write any vector as a linear combination of these vectors.

Matrix Representation Theorem

If a vector space has a basis $e_1, e_2, \ldots$, we can write any vector as a linear combination of these vectors.
This implies that $T(\overrightarrow{v}) = T(a_1 e_1 + a_2 e_2 + \ldots) = a_1 T(e_1) + a_2 T(e_2) + \ldots$ for some collection of scalars $a_1, a_2, \ldots$

Matrix Representation Theorem

CPMSOC

If a vector space has a basis $e_1, e_2, \ldots$, we can write any vector as a linear combination of these vectors.
This implies that $T(\overrightarrow{v}) = T(a_1e_1 + a_2e_2 + \ldots) = a_1T(e_1) + a_2T(e_2) + \ldots$ for some collection of scalars $a_1, a_2, \ldots$
Let $e_i' = T(e_i)$;

Matrix Representation Theorem

CPMSOC

If a vector space has a basis $e_1, e_2, \ldots$, we can write any vector as a linear combination of these vectors.
This implies that $T(\vec{v}) = T(a_1e_1 + a_2e_2 + \ldots) = a_1T(e_1) + a_2T(e_2) + \ldots$ for some collection of scalars $a_1, a_2, \ldots$
Let $e_i' = T(e_i)$; then the matrix product

$$(e_1'|e_2'|\ldots) \cdot \begin{pmatrix} a_1 \\ a_2 \\ \vdots \end{pmatrix}$$

gives the coordinates of $T(\vec{v})$ in terms of the chosen basis.

Matrix Representation Theorem

If a vector space has a basis $e_1, e_2, \ldots$, we can write any vector as a linear combination of these vectors.
This implies that $T(\overrightarrow{v}) = T(a_1e_1 + a_2e_2 + \ldots) = a_1T(e_1) + a_2T(e_2) + \ldots$ for some collection of scalars $a_1, a_2, \ldots$
Let $e_i' = T(e_i)$; then the matrix product

$$(e_1'|e_2'|\ldots) \cdot \begin{pmatrix} a_1 \\ a_2 \\ \vdots \end{pmatrix}$$

gives the coordinates of $T(\overrightarrow{v})$ in terms of the chosen basis.
That is, for every vector space with a basis (e.g. $span\{sin(x), cos(x)\}$), a linear mapping from that vector space to itself (e.g. $\frac{d}{dx}$) can be represented as a matrix with dimension equal to that of the vector space (e.g. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$).

Matrix Representation Theorem

CPMSOC

If a vector space has a basis $e_1, e_2, \ldots$, we can write any vector as a linear combination of these vectors.
This implies that $T(\overrightarrow{v}) = T(a_1 e_1 + a_2 e_2 + \ldots) = a_1 T(e_1) + a_2 T(e_2) + \ldots$ for some collection of scalars $a_1, a_2, \ldots$
Let $e_i' = T(e_i)$; then the matrix product

$$(e_1'|e_2'|\ldots) \cdot \begin{pmatrix} a_1 \\ a_2 \\ \vdots \end{pmatrix}$$

gives the coordinates of $T(\overrightarrow{v})$ in terms of the chosen basis.
That is, for every vector space with a basis (e.g. $span\{sin(x), cos(x)\}$), a linear mapping from that vector space to itself (e.g. $\frac{d}{dx}$) can be represented as a matrix with dimension equal to that of the vector space (e.g. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$).
Matrix multiplication in general is defined as $(AB)_{i,j} = \sum_k A_{i,k} B_{k,j}$.

Applications of Matrices

Why do we care about matrices?

Example

Let F_n denote the size of a rabbit population in month n, such that $F_1 = F_2 = 1, F_{n+2} = F_{n+1} + F_n$.
Show that $F_{m+n+1} = F_{m+1}F_{n+1} + F_mF_n$.

Consider the matrix $M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

Applications of Matrices

Why do we care about matrices?

Example

Let F_n denote the size of a rabbit population in month n, such that $F_1 = F_2 = 1, F_{n+2} = F_{n+1} + F_n$.
Show that $F_{m+n+1} = F_{m+1}F_{n+1} + F_mF_n$.

Consider the matrix $M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

An inductive argument shows that $M^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$.

Applications of Matrices

Why do we care about matrices?

Example

Let F_n denote the size of a rabbit population in month n, such that $F_1 = F_2 = 1, F_{n+2} = F_{n+1} + F_n$.
Show that $F_{m+n+1} = F_{m+1}F_{n+1} + F_mF_n$.

Consider the matrix $M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

An inductive argument shows that $M^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$.

Notice that $M^{m+n} = M^m M^n$ by associativity, i.e.
$\begin{pmatrix} F_{m+n+1} & F_{m+n} \\ F_{m+n} & F_{m+n-1} \end{pmatrix} = \begin{pmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{pmatrix} \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$.

Applications of Matrices

Why do we care about matrices?

Example

Let F_n denote the size of a rabbit population in month n, such that $F_1 = F_2 = 1, F_{n+2} = F_{n+1} + F_n$.
Show that $F_{m+n+1} = F_{m+1}F_{n+1} + F_mF_n$.

Consider the matrix $M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

An inductive argument shows that $M^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$.

Notice that $M^{m+n} = M^m M^n$ by associativity, i.e.

$$\begin{pmatrix} F_{m+n+1} & F_{m+n} \\ F_{m+n} & F_{m+n-1} \end{pmatrix} = \begin{pmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{pmatrix} \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}.$$

Since $(AB)_{i,j} = \sum_k A_{i,k}B_{k,j}$, comparing top-left entries demonstrates the desired identity.

Determinants

For an $n \times n$ matrix A, we compute $\det(A)$ using the recursive formula

$$|A| = \det(A) = \sum_{1 \le i \le n} (-1)^{i+1} A_{1,i} \det(A \backslash (1, \cdot) \backslash (\cdot, i)).$$

Determinants

For an $n \times n$ matrix A, we compute $\det(A)$ using the recursive formula

$$|A| = \det(A) = \sum_{1 \leq i \leq n} (-1)^{i+1} A_{1,i} \det(A \backslash (1, \cdot) \backslash (\cdot, i)).$$

For example,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

Determinants

For an $n \times n$ matrix A, we compute $\det(A)$ using the recursive formula

$$|A| = \det(A) = \sum_{1 \le i \le n} (-1)^{i+1} A_{1,i} \det(A \backslash (1,\cdot) \backslash (\cdot, i)).$$

For example,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

A matrix is invertible if and only if its determinant is nonzero.

Determinants

CPMSOC

Example (Vandermonde Determinants)

Let $x_1, x_2, \dots x_n$ be arbitrary numbers for some $n \geq 1$.
Computer the determinant

$$\det(V_n) = \begin{vmatrix} x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & \dots & x_n \\ 1 & 1 & \dots & 1 \end{vmatrix}.$$

Determinants

CPMSOC

Example (Vandermonde Determinants)

Let $x_1, x_2, \dots x_n$ be arbitrary numbers for some $n \geq 1$.
Computer the determinant

$$\det(V_n) = \begin{vmatrix} x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & \dots & x_n \\ 1 & 1 & \dots & 1 \end{vmatrix}.$$

Notice that the determinant will be an $(n-1)$-degree polynomial in x_n.

Determinants

CPMSOC

Example (Vandermonde Determinants)

Let $x_1, x_2, \dots x_n$ be arbitrary numbers for some $n \geq 1$.
Computer the determinant

$$\det(V_n) = \begin{vmatrix} x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & \dots & x_n \\ 1 & 1 & \dots & 1 \end{vmatrix}.$$

Notice that the determinant will be an $(n-1)$-degree polynomial in x_n.
The roots will be $x_1, x_2, \dots x_{n-1}$

Determinants

CPMSOC

ARC CLUBS SUPPORTED BY Arc INDEPENDENTLY RUN

Example (Vandermonde Determinants)

Let $x_1, x_2, \ldots x_n$ be arbitrary numbers for some $n \geq 1$.
Computer the determinant

$$\det(V_n) = \begin{vmatrix} x_1^{n-1} & x_2^{n-1} & \ldots & x_n^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & \ldots & x_n \\ 1 & 1 & \ldots & 1 \end{vmatrix}.$$

Notice that the determinant will be an $(n-1)$-degree polynomial in x_n.
The roots will be $x_1, x_2, \ldots x_{n-1}$ and the leading coefficient will be a Vandermonde determinant of order $(n-1)$, multiplied by $(-1)^{n+1}$.

Determinants

CPMSOC

Example (Vandermonde Determinants)

Let $x_1, x_2, \dots x_n$ be arbitrary numbers for some $n \geq 1$.
Computer the determinant

$$\det(V_n) = \begin{vmatrix} x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & \dots & x_n \\ 1 & 1 & \dots & 1 \end{vmatrix}.$$

Notice that the determinant will be an $(n-1)$-degree polynomial in x_n.
The roots will be $x_1, x_2, \dots x_{n-1}$ and the leading coefficient will be a Vandermonde determinant of order $(n-1)$, multiplied by $(-1)^{n+1}$.
In other words, $\det(V_n) = (-1)^{n+1}\det(V_{n-1})(x_n - x_1)(x_n - x_2)\dots(x_n - x_{n-1})$.
An inductive argument shows that $\det(V_n) = \prod_{i>j}(x_i - x_j)$.

Inversion

CPMSOC

Example

Let A be an $n \times n$ symmetric matrix (i.e., $A = A^T$) with positive real entries, for some $n \geq 2$.
Show that A^{-1} has at most $n^2 - 2n$ entries equal to zero.

Observe that $\sum_k A_{i,k} A^{-1}_{k,j} = \delta_{i,j}$.

Inversion

Example

Let A be an $n \times n$ symmetric matrix (i.e., $A = A^T$) with positive real entries, for some $n \geq 2$.
Show that A^{-1} has at most $n^2 - 2n$ entries equal to zero.

Observe that $\sum_k A_{i,k} A^{-1}_{k,j} = \delta_{i,j}$. Since A is positive, A^{-1} has at least one positive and at least one negative entry for each value of j.

Inversion

CPMSOC

Example

Let A be an $n \times n$ symmetric matrix (i.e., $A = A^T$) with positive real entries, for some $n \geq 2$.
Show that A^{-1} has at most $n^2 - 2n$ entries equal to zero.

Observe that $\sum_k A_{i,k} A^{-1}_{k,j} = \delta_{i,j}$. Since A is positive, A^{-1} has at least one positive and at least one negative entry for each value of j.
Hence every column of A^{-1} contains at least two nonzero entries. This implies the desired result.

Independence

Let's generalise what we mean by independence.

Take a set $\mathbb{V}$. We'll call some of its subsets *independent* - in particular, $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2, \ldots$

We require the following properties:

Independence

Let's generalise what we mean by independence.
Take a set $\mathbb{V}$. We'll call some of its subsets *independent* - in particular, $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2, \ldots$
We require the following properties:

- The empty set is independent

Independence

Let's generalise what we mean by independence.

Take a set $\mathbb{V}$. We'll call some of its subsets *independent* - in particular, $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2, \ldots$

We require the following properties:

- The empty set is independent
- Every subset of an independent set is independent

Independence

Let's generalise what we mean by independence.

Take a set $\mathbb{V}$. We'll call some of its subsets *independent* - in particular, $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2, \ldots$

We require the following properties:

- The empty set is independent
- Every subset of an independent set is independent
- If A and B are independent sets, and A is bigger than B, then there's some element in A we can add to B to get a new independent set

Independence

Let's generalise what we mean by independence.
Take a set $\mathbb{V}$. We'll call some of its subsets *independent* - in particular, $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2, \ldots$
We require the following properties:

- The empty set is independent
- Every subset of an independent set is independent
- If A and B are independent sets, and A is bigger than B, then there's some element in A we can add to B to get a new independent set

The dimension is the largest number of elements in an independent set.
If a set $\mathbb{V}$ and subsets $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2, \ldots$ satisfy these properties, we call them a *matroid*.

Rank

The row rank of M is the dimension of the span of its rows.
The column rank of M is defined similarly.

Rank

The row rank of M is the dimension of the span of its rows.
The column rank of M is defined similarly.
These are always equal.

Rank

The row rank of M is the dimension of the span of its rows.
The column rank of M is defined similarly.
These are always equal. In the special case of invertible matrices, we can see this by reducing them to RREF (since the required row operations will not change the rank), and observing that the entries off the diagonal are all zero.

$$RREF = \begin{pmatrix} a_1 & 0 & 0 & \dots \\ 0 & a_2 & 0 & \dots \\ 0 & 0 & a_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Independence

Example

Let Z denote the set of points in $\mathbb{R}^n$ whose coordinates are 0 or 1. Let k be a fixed number between 0 and n. If V is a k-dimensional subspace of $\mathbb{R}^n$, find the maximum possible number of points in $Z \cap V$.

Consider the matrix M whose rows are the elements of $Z \cap V$. By construction, the span of the rows (and therefore of the columns) is at most k-dimensional.

Independence

Example

Let Z denote the set of points in $\mathbb{R}^n$ whose coordinates are 0 or 1. Let k be a fixed number between 0 and n. If V is a k-dimensional subspace of $\mathbb{R}^n$, find the maximum possible number of points in $Z \cap V$.

Consider the matrix M whose rows are the elements of $Z \cap V$. By construction, the span of the rows (and therefore of the columns) is at most k-dimensional.
From the columns of M, choose a set of k columns which spans the column space.

Independence

Example

Let Z denote the set of points in $\mathbb{R}^n$ whose coordinates are 0 or 1. Let k be a fixed number between 0 and n. If V is a k-dimensional subspace of $\mathbb{R}^n$, find the maximum possible number of points in $Z \cap V$.

Consider the matrix M whose rows are the elements of $Z \cap V$. By construction, the span of the rows (and therefore of the columns) is at most k-dimensional.
From the columns of M, choose a set of k columns which spans the column space.
For each row, we only need to know the coordinates in these k different positions to infer the rest of the coordinates.

Independence

Example

Let Z denote the set of points in $\mathbb{R}^n$ whose coordinates are 0 or 1. Let k be a fixed number between 0 and n. If V is a k-dimensional subspace of $\mathbb{R}^n$, find the maximum possible number of points in $Z \cap V$.

Consider the matrix M whose rows are the elements of $Z \cap V$. By construction, the span of the rows (and therefore of the columns) is at most k-dimensional.
From the columns of M, choose a set of k columns which spans the column space.
For each row, we only need to know the coordinates in these k different positions to infer the rest of the coordinates.
There are therefore at most 2^k rows.

Independence

Example

Let Z denote the set of points in $\mathbb{R}^n$ whose coordinates are 0 or 1. Let k be a fixed number between 0 and n. If V is a k-dimensional subspace of $\mathbb{R}^n$, find the maximum possible number of points in $Z \cap V$.

Consider the matrix M whose rows are the elements of $Z \cap V$. By construction, the span of the rows (and therefore of the columns) is at most k-dimensional.
From the columns of M, choose a set of k columns which spans the column space.
For each row, we only need to know the coordinates in these k different positions to infer the rest of the coordinates.
There are therefore at most 2^k rows.
Now take V to be the span of the set of vectors having all possible entries in the first k rows, and zero thereafter.

Independence

Example

Let Z denote the set of points in $\mathbb{R}^n$ whose coordinates are 0 or 1. Let k be a fixed number between 0 and n. If V is a k-dimensional subspace of $\mathbb{R}^n$, find the maximum possible number of points in $Z \cap V$.

Consider the matrix M whose rows are the elements of $Z \cap V$. By construction, the span of the rows (and therefore of the columns) is at most k-dimensional.
From the columns of M, choose a set of k columns which spans the column space.
For each row, we only need to know the coordinates in these k different positions to infer the rest of the coordinates.
There are therefore at most 2^k rows.
Now take V to be the span of the set of vectors having all possible entries in the first k rows, and zero thereafter. It is clear that $Z \cap V$ has 2^k points, so this is the maximum possible number.

Independence

Example

Let X and B_0 be $n \times n$ matrices, $n \geq 1$. Define $B_i = B_{i-1}X - XB_{i-1}$, for $i \geq 1$. Show that if $X = B_{n^2}$, then X must be the zero matrix, $\mathcal{O}_n$.

Notice that $B_{n^2+1} = B_{n^2}X - XB_{n^2} = X^2 - X^2 = \mathcal{O}_n$, and similarly for higher indices $n^2 + j$.

Independence

CPMSOC

Example

Let X and B_0 be $n \times n$ matrices, $n \geq 1$. Define $B_i = B_{i-1}X - XB_{i-1}$, for $i \geq 1$. Show that if $X = B_{n^2}$, then X must be the zero matrix, $\mathcal{O}_n$.

Notice that $B_{n^2+1} = B_{n^2}X - XB_{n^2} = X^2 - X^2 = \mathcal{O}_n$, and similarly for higher indices $n^2 + j$.
Since the space of $n \times n$ matrices is n^2-dimensional, the matrices $B_0, B_1, \ldots B_{n^2}$ must be linearly dependent,

Independence

CPMSOC

Example

Let X and B_0 be $n \times n$ matrices, $n \geq 1$. Define $B_i = B_{i-1}X - XB_{i-1}$, for $i \geq 1$. Show that if $X = B_{n^2}$, then X must be the zero matrix, $\mathcal{O}_n$.

Notice that $B_{n^2+1} = B_{n^2}X - XB_{n^2} = X^2 - X^2 = \mathcal{O}_n$, and similarly for higher indices $n^2 + j$.
Since the space of $n \times n$ matrices is n^2-dimensional, the matrices $B_0, B_1, \ldots B_{n^2}$ must be linearly dependent, i.e. we can choose scalars such that $c_0 B_0 + c_1 B_1 + \ldots c_{n^2} B_{n^2} = \mathcal{O}_n$.
Pick the first k such that $c_k \neq 0$.

Independence

Example

Let X and B_0 be $n \times n$ matrices, $n \geq 1$. Define $B_i = B_{i-1}X - XB_{i-1}$, for $i \geq 1$. Show that if $X = B_{n^2}$, then X must be the zero matrix, $\mathcal{O}_n$.

Notice that $B_{n^2+1} = B_{n^2}X - XB_{n^2} = X^2 - X^2 = \mathcal{O}_n$, and similarly for higher indices $n^2 + j$.
Since the space of $n \times n$ matrices is n^2-dimensional, the matrices $B_0, B_1, \ldots B_{n^2}$ must be linearly dependent, i.e. we can choose scalars such that $c_0B_0 + c_1B_1 + \ldots c_{n^2}B_{n^2} = \mathcal{O}_n$.
Pick the first k such that $c_k \neq 0$. Then $-c_kB_k = c_{k+1}B_{k+1} + \ldots + c_{n^2}B_{n^2}$.

Independence

CPMSOC

Example

Let X and B_0 be $n \times n$ matrices, $n \geq 1$. Define $B_i = B_{i-1}X - XB_{i-1}$, for $i \geq 1$. Show that if $X = B_{n^2}$, then X must be the zero matrix, $\mathcal{O}_n$.

Notice that $B_{n^2+1} = B_{n^2}X - XB_{n^2} = X^2 - X^2 = \mathcal{O}_n$, and similarly for higher indices $n^2 + j$.
Since the space of $n \times n$ matrices is n^2-dimensional, the matrices $B_0, B_1, \ldots B_{n^2}$ must be linearly dependent, i.e. we can choose scalars such that $c_0B_0 + c_1B_1 + \ldots c_{n^2}B_{n^2} = \mathcal{O}_n$.
Pick the first k such that $c_k \neq 0$. Then $-c_kB_k = c_{k+1}B_{k+1} + \ldots + c_{n^2}B_{n^2}$.
Applying the rule,
$-c_kB_{k+1} = -(c_kB_kX - c_kXB_k) = c_{k+1}B_{k+1}X + \ldots + c_nB_nX - c_{k+1}XB_{k+1} - \ldots - c_{n^2}XB_{n^2}$.

Independence

CPMSOC

Example

Let X and B_0 be $n \times n$ matrices, $n \geq 1$. Define $B_i = B_{i-1}X - XB_{i-1}$, for $i \geq 1$. Show that if $X = B_{n^2}$, then X must be the zero matrix, $\mathcal{O}_n$.

Notice that $B_{n^2+1} = B_{n^2}X - XB_{n^2} = X^2 - X^2 = \mathcal{O}_n$, and similarly for higher indices $n^2 + j$.
Since the space of $n \times n$ matrices is n^2-dimensional, the matrices $B_0, B_1, \ldots B_{n^2}$ must be linearly dependent, i.e. we can choose scalars such that $c_0B_0 + c_1B_1 + \ldots c_{n^2}B_{n^2} = \mathcal{O}_n$.
Pick the first k such that $c_k \neq 0$. Then $-c_kB_k = c_{k+1}B_{k+1} + \ldots + c_{n^2}B_{n^2}$.
Applying the rule,
$-c_kB_{k+1} = -(c_kB_kX - c_kXB_k) = c_{k+1}B_{k+1}X + \ldots + c_nB_nX - c_{k+1}XB_{k+1} - \ldots - c_{n^2}XB_{n^2}$.
That is, $-c_kB_{k+1} = c_{k+1}B_{k+2} + \ldots + c_{n^2}B_{n^2+1}$.

Independence

CPMSOC

Example

Let X and B_0 be $n \times n$ matrices, $n \geq 1$. Define $B_i = B_{i-1}X - XB_{i-1}$, for $i \geq 1$. Show that if $X = B_{n^2}$, then X must be the zero matrix, $\mathcal{O}_n$.

Notice that $B_{n^2+1} = B_{n^2}X - XB_{n^2} = X^2 - X^2 = \mathcal{O}_n$, and similarly for higher indices $n^2 + j$.
$-c_k B_k = c_{k+1}B_{k+1} + \ldots + c_{n^2}B_{n^2}$.
Applying the rule,
$-c_k B_{k+1} = -(c_k B_k X - c_k X B_k) = c_{k+1}B_{k+1}X + \ldots + c_n B_n X - c_{k+1}XB_{k+1} - \ldots - c_{n^2}XB_{n^2}$.
That is, $-c_k B_{k+1} = c_{k+1}B_{k+2} + \ldots + c_{n^2}B_{n^2+1}$.

Independence

Example

Let X and B_0 be $n \times n$ matrices, $n \geq 1$. Define $B_i = B_{i-1}X - XB_{i-1}$, for $i \geq 1$. Show that if $X = B_{n^2}$, then X must be the zero matrix, $\mathcal{O}_n$.

Notice that $B_{n^2+1} = B_{n^2}X - XB_{n^2} = X^2 - X^2 = \mathcal{O}_n$, and similarly for higher indices $n^2 + j$.

$-c_k B_k = c_{k+1}B_{k+1} + \ldots + c_{n^2}B_{n^2}$.

Applying the rule,

$-c_k B_{k+1} = -(c_k B_k X - c_k X B_k) = c_{k+1}B_{k+1}X + \ldots + c_n B_n X - c_{k+1}XB_{k+1} - \ldots - c_{n^2}XB_{n^2}$.

That is, $-c_k B_{k+1} = c_{k+1}B_{k+2} + \ldots + c_{n^2}B_{n^2+1}$.

By an inductive argument, $-c_k B_{k+j} = c_{k+1}B_{k+1+j} + \ldots + c_{n^2}B_{n^2+j}$.

Independence

Example

Let X and B_0 be $n \times n$ matrices, $n \geq 1$. Define $B_i = B_{i-1}X - XB_{i-1}$, for $i \geq 1$. Show that if $X = B_{n^2}$, then X must be the zero matrix, $\mathcal{O}_n$.

Notice that $B_{n^2+1} = B_{n^2}X - XB_{n^2} = X^2 - X^2 = \mathcal{O}_n$, and similarly for higher indices $n^2 + j$.

$-c_k B_k = c_{k+1}B_{k+1} + \ldots + c_{n^2}B_{n^2}$.

Applying the rule,

$-c_k B_{k+1} = -(c_k B_k X - c_k X B_k) = c_{k+1}B_{k+1}X + \ldots + c_n B_n X - c_{k+1}XB_{k+1} - \ldots - c_{n^2}XB_{n^2}$.

That is, $-c_k B_{k+1} = c_{k+1}B_{k+2} + \ldots + c_{n^2}B_{n^2+1}$.

By an inductive argument, $-c_k B_{k+j} = c_{k+1}B_{k+1+j} + \ldots + c_{n^2}B_{n^2+j}$.

In particular, $-c_k B_{n^2} = c_{k+1}B_{n^2+1} + \ldots + c_{n^2}B_{n^2+k}$

Independence

CPMSOC

Example

Let X and B_0 be $n \times n$ matrices, $n \geq 1$. Define $B_i = B_{i-1}X - XB_{i-1}$, for $i \geq 1$. Show that if $X = B_{n^2}$, then X must be the zero matrix, $\mathcal{O}_n$.

Notice that $B_{n^2+1} = B_{n^2}X - XB_{n^2} = X^2 - X^2 = \mathcal{O}_n$, and similarly for higher indices $n^2 + j$.

$-c_k B_k = c_{k+1}B_{k+1} + \ldots + c_{n^2}B_{n^2}$.

Applying the rule,

$-c_k B_{k+1} = -(c_k B_k X - c_k X B_k) = c_{k+1}B_{k+1}X + \ldots + c_n B_n X - c_{k+1}XB_{k+1} - \ldots - c_{n^2}XB_{n^2}$.

That is, $-c_k B_{k+1} = c_{k+1}B_{k+2} + \ldots + c_{n^2}B_{n^2+1}$.

By an inductive argument, $-c_k B_{k+j} = c_{k+1}B_{k+1+j} + \ldots + c_{n^2}B_{n^2+j}$.

In particular, $-c_k B_{n^2} = c_{k+1}B_{n^2+1} + \ldots + c_{n^2}B_{n^2+k}$

So $-c_k X = c_{k+1}\mathcal{O}_n + \ldots + c_{n^2}\mathcal{O}_n = \mathcal{O}_n$.

Eigenvalues and Eigenvectors

Consider the matrix equation $A\overrightarrow{v} = \lambda\overrightarrow{v}$, for some fixed A.

Eigenvalues and Eigenvectors

Consider the matrix equation $A\overrightarrow{v} = \lambda\overrightarrow{v}$, for some fixed A.
Possible values of $\overrightarrow{v} \in \mathbb{C}^n$ are called eigenvectors, and possible values of $\lambda \in \mathbb{C}^n$ are called eigenvalues.

Eigenvalues and Eigenvectors

Consider the matrix equation $A\overrightarrow{v} = \lambda\overrightarrow{v}$, for some fixed A.
Possible values of $\overrightarrow{v} \in \mathbb{C}^n$ are called eigenvectors, and possible values of $\lambda \in \mathbb{C}^n$ are called eigenvalues.
To find the eigenvalues, we solve the *characteristic equation* $\det(\lambda I_n - A) = 0$, which is a polynomial in λ.

Eigenvalues and Eigenvectors

Consider the matrix equation $A\overrightarrow{v} = \lambda\overrightarrow{v}$, for some fixed A.
Possible values of $\overrightarrow{v} \in \mathbb{C}^n$ are called eigenvectors, and possible values of $\lambda \in \mathbb{C}^n$ are called eigenvalues.
To find the eigenvalues, we solve the *characteristic equation* $\det(\lambda I_n - A) = 0$, which is a polynomial in λ.
In the case where these eigenvalues are each roots of multiplicity one, the matrix A can be transformed to a diagonal matrix with a change of basis: $A = BDB^{-1}$.

Eigenvalues and Eigenvectors

Consider the matrix equation $A\overrightarrow{v} = \lambda\overrightarrow{v}$, for some fixed A.
Possible values of $\overrightarrow{v} \in \mathbb{C}^n$ are called eigenvectors, and possible values of $\lambda \in \mathbb{C}^n$ are called eigenvalues.
To find the eigenvalues, we solve the *characteristic equation* $\det(\lambda I_n - A) = 0$, which is a polynomial in λ.
In the case where these eigenvalues are each roots of multiplicity one, the matrix A can be transformed to a diagonal matrix with a change of basis: $A = BDB^{-1}$. The new basis is called an *eigenbasis*, and consists of the eigenvectors of the matrix.

Eigenvalues and Eigenvectors

CPMSOC

Example

Let $A : V \to W$ and $B : W \to V$ be linear maps between finite-dimensional vector spaces. Prove that the linear maps AB and BA have the same set of nonzero eigenvalues, counted with multiplicity.

Choose a basis that identifies V with $\mathbb{R}^m$ and W with $\mathbb{R}^n$.

Eigenvalues and Eigenvectors

CPMSOC

Example

Let $A : V \to W$ and $B : W \to V$ be linear maps between finite-dimensional vector spaces. Prove that the linear maps AB and BA have the same set of nonzero eigenvalues, counted with multiplicity.

Choose a basis that identifies V with $\mathbb{R}^m$ and W with $\mathbb{R}^n$. Now represent A and B as matrices and notice that our claim is equivalent to $\det(\lambda \mathcal{I}_n - AB) = \lambda^{n-m} \det(\lambda \mathcal{I}_m - BA)$.

Eigenvalues and Eigenvectors

CPMSOC

Example

Let $A : V \to W$ and $B : W \to V$ be linear maps between finite-dimensional vector spaces. Prove that the linear maps AB and BA have the same set of nonzero eigenvalues, counted with multiplicity.

Choose a basis that identifies V with $\mathbb{R}^m$ and W with $\mathbb{R}^n$. Now represent A and B as matrices and notice that our claim is equivalent to $\det(\lambda\mathcal{I}_n - AB) = \lambda^{n-m}\det(\lambda\mathcal{I}_m - BA)$. Assume WLOG that $n \geq m$.

Eigenvalues and Eigenvectors

Example

Let $A : V \to W$ and $B : W \to V$ be linear maps between finite-dimensional vector spaces. Prove that the linear maps AB and BA have the same set of nonzero eigenvalues, counted with multiplicity.

Choose a basis that identifies V with $\mathbb{R}^m$ and W with $\mathbb{R}^n$. Now represent A and B as matrices and notice that our claim is equivalent to $\det(\lambda\mathcal{I}_n - AB) = \lambda^{n-m}\det(\lambda\mathcal{I}_m - BA)$. Assume WLOG that $n \geq m$. Now transform A, B to $n \times n$ matrices A', B'.

Eigenvalues and Eigenvectors

Example

Let $A : V \to W$ and $B : W \to V$ be linear maps between finite-dimensional vector spaces. Prove that the linear maps AB and BA have the same set of nonzero eigenvalues, counted with multiplicity.

Choose a basis that identifies V with $\mathbb{R}^m$ and W with $\mathbb{R}^n$. Now represent A and B as matrices and notice that our claim is equivalent to $\det(\lambda\mathcal{I}_n - AB) = \lambda^{n-m}\det(\lambda\mathcal{I}_m - BA)$. Assume WLOG that $n \geq m$. Now transform A, B to $n \times n$ matrices A', B'. Since $\det(\lambda\mathcal{I}_n - A'B') = \det(\lambda\mathcal{I}_n - AB)$ and $\det(\lambda\mathcal{I}_n - B'A') = \lambda^{n-m}\det(\lambda\mathcal{I}_n - BA)$,

Eigenvalues and Eigenvectors

CPMSOC

Example

Let $A : V \to W$ and $B : W \to V$ be linear maps between finite-dimensional vector spaces. Prove that the linear maps AB and BA have the same set of nonzero eigenvalues, counted with multiplicity.

Choose a basis that identifies V with $\mathbb{R}^m$ and W with $\mathbb{R}^n$. Now represent A and B as matrices and notice that our claim is equivalent to $\det(\lambda\mathcal{I}_n - AB) = \lambda^{n-m}\det(\lambda\mathcal{I}_m - BA)$. Assume WLOG that $n \geq m$. Now transform A, B to $n \times n$ matrices A', B'. Since $\det(\lambda\mathcal{I}_n - A'B') = \det(\lambda\mathcal{I}_n - AB)$ and $\det(\lambda\mathcal{I}_n - B'A') = \lambda^{n-m}\det(\lambda\mathcal{I}_n - BA)$, we simply require that $\det(\lambda\mathcal{I}_n - A'B') = \det(\lambda\mathcal{I}_n - B'A')$.

Eigenvalues and Eigenvectors

CPMSOC

Example

Let $A : V \to W$ and $B : W \to V$ be linear maps between finite-dimensional vector spaces. Prove that the linear maps AB and BA have the same set of nonzero eigenvalues, counted with multiplicity.

Choose a basis that identifies V with $\mathbb{R}^m$ and W with $\mathbb{R}^n$. Now represent A and B as matrices and notice that our claim is equivalent to $\det(\lambda\mathcal{I}_n - AB) = \lambda^{n-m}\det(\lambda\mathcal{I}_m - BA)$. Assume WLOG that $n \geq m$. Now transform A, B to $n \times n$ matrices A', B'. Since $\det(\lambda\mathcal{I}_n - A'B') = \det(\lambda\mathcal{I}_n - AB)$ and $\det(\lambda\mathcal{I}_n - B'A') = \lambda^{n-m}\det(\lambda\mathcal{I}_n - BA)$, we simply require that $\det(\lambda\mathcal{I}_n - A'B') = \det(\lambda\mathcal{I}_n - B'A')$. This is true in general for $n \times n$ matrices.

Spectral Mapping Theorem

In physics, the set of eigenvectors of a matrix is often called its spectrum.

Theorem (Spectral Mapping Theorem)

Let A be a matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots$, and $P(x)$ be a polynomial.
Then the eigenvalues of $P(A)$ are $P(\lambda_1), P(\lambda_2), P(\lambda_3), \ldots$

To prove this for the case where eigenvalues are distinct, we notice that A is a diagonal matrix with respect to the eigenbasis, such that

$$P(A) = P\left(\begin{pmatrix} \lambda_1 & 0 & \cdots \\ 0 & \lambda_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}\right) = \begin{pmatrix} P(\lambda_1) & 0 & \cdots \\ 0 & P(\lambda_2) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

, which has eigenvalues as specified.

Spectral Mapping Theorem

CPMSOC

In physics, the set of eigenvectors of a matrix is often called its spectrum.

Theorem (Spectral Mapping Theorem)

Let A be a matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots$, and $P(x)$ be a polynomial. Then the eigenvalues of $P(A)$ are $P(\lambda_1), P(\lambda_2), P(\lambda_3), \ldots$

To prove this for the case where eigenvalues are distinct, we notice that A is a diagonal matrix with respect to the eigenbasis, such that

$$P(A) = P\left(\begin{pmatrix} \lambda_1 & 0 & \cdots \\ 0 & \lambda_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}\right) = \begin{pmatrix} P(\lambda_1) & 0 & \cdots \\ 0 & P(\lambda_2) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

, which has eigenvalues as specified. Since the characteristic polynomial depends continuously on the entries, and the roots of this polynomial depend continuously on its coefficients, we can "nudge" any non-distinct eigenvalues and use a limiting argument to show that this is true in the general case.

Cayley-Hamilton Theorem

CPMSOC

Theorem (Cayley-Hamilton Theorem)

Every square matrix is a root of its characteristic polynomial.

Cayley-Hamilton Theorem

Theorem (Cayley-Hamilton Theorem)

Every square matrix is a root of its characteristic polynomial.

Example

Let A and B be 2×2 matrices with determinant equal to 1. Prove that $\mathrm{tr}(AB)-(\mathrm{tr}A)(\mathrm{tr}B)+\mathrm{tr}(AB^{-1})=0$.

Cayley-Hamilton Theorem

CPMSOC

Arc CLUBS SUPPORTED BY Arc INDEPENDENTLY RUN

Theorem (Cayley-Hamilton Theorem)

Every square matrix is a root of its characteristic polynomial.

Example

Let A and B be 2×2 matrices with determinant equal to 1. Prove that $\text{tr}(AB) - (\text{tr}A)(\text{tr}B) + \text{tr}(AB^{-1}) = 0$.

The characteristic polynomial of B is given by $\det(\lambda \mathcal{I}_2 - B) = \begin{vmatrix} \lambda - B_{1,1} & B_{1,2} \\ B_{2,1} & \lambda - B_{2,2} \end{vmatrix} = (\lambda - B_{1,1})(\lambda - B_{2,2}) - B_{1,2}B_{2,1} = \lambda^2 - tr(B)\lambda + \det B = \lambda^2 - tr(B) + 1$.

Cayley-Hamilton Theorem

CPMSOC

Theorem (Cayley-Hamilton Theorem)

Every square matrix is a root of its characteristic polynomial.

Example

Let A and B be 2×2 matrices with determinant equal to 1. Prove that tr$(AB)-($tr$A)($tr$B)+$ tr$(AB^{-1})=0$.

The characteristic polynomial of B is given by $\det(\lambda\mathcal{I}_2-B)=\begin{vmatrix}\lambda-B_{1,1} & B_{1,2}\\ B_{2,1} & \lambda-B_{2,2}\end{vmatrix}=$ $(\lambda-B_{1,1})(\lambda-B_{2,2})-B_{1,2}B_{2,1}=\lambda^2-tr(B)\lambda+\det B=\lambda^2-tr(B)+1$.
By the Cayley-Hamilton theorem, $B^2-tr(B)B+\mathcal{I}_2=\mathcal{O}_2$.

Cayley-Hamilton Theorem

CPMSOC

Theorem (Cayley-Hamilton Theorem)

Every square matrix is a root of its characteristic polynomial.

Example

Let A and B be 2×2 matrices with determinant equal to 1. Prove that tr$(AB)-($tr$A)($tr$B)+$ tr$(AB^{-1})=0$.

The characteristic polynomial of B is given by $\det(\lambda\mathcal{I}_2 - B) = \begin{vmatrix} \lambda - B_{1,1} & B_{1,2} \\ B_{2,1} & \lambda - B_{2,2} \end{vmatrix} =$ $(\lambda - B_{1,1})(\lambda - B_{2,2}) - B_{1,2}B_{2,1} = \lambda^2 - tr(B)\lambda + \det B = \lambda^2 - tr(B) + 1$.
By the Cayley-Hamilton theorem, $B^2 - tr(B)B + \mathcal{I}_2 = \mathcal{O}_2$.
Multiply by AB^{-1} to obtain $AB - tr(B)A + AB^{-1} = \mathcal{O}_2$.

Cayley-Hamilton Theorem

CPMSOC

Arc CLUBS SUPPORTED BY Arc INDEPENDENTLY RUN

Theorem (Cayley-Hamilton Theorem)

Every square matrix is a root of its characteristic polynomial.

Example

Let A and B be 2×2 matrices with determinant equal to 1. Prove that tr$(AB)-($tr$A)($tr$B)+$ tr$(AB^{-1})=0$.

The characteristic polynomial of B is given by $\det(\lambda\mathcal{I}_2 - B) = \begin{vmatrix} \lambda - B_{1,1} & B_{1,2} \\ B_{2,1} & \lambda - B_{2,2} \end{vmatrix} =$ $(\lambda - B_{1,1})(\lambda - B_{2,2}) - B_{1,2}B_{2,1} = \lambda^2 - tr(B)\lambda + \det B = \lambda^2 - tr(B) + 1$.
By the Cayley-Hamilton theorem, $B^2 - tr(B)B + \mathcal{I}_2 = \mathcal{O}_2$.
Multiply by AB^{-1} to obtain $AB - tr(B)A + AB^{-1} = \mathcal{O}_2$.
Taking the trace of both sides gives the desired result.

Cayley-Hamilton Theorem

Theorem (Cayley-Hamilton Theorem)

Every square matrix is a root of its characteristic polynomial.

Example

Let A and B be 2×2 matrices with determinant equal to 1. Prove that tr$(AB)-($tr$A)($tr$B)+$ tr$(AB^{-1})=0$.

The characteristic polynomial of B is given by $\det(\lambda\mathcal{I}_2 - B) = \begin{vmatrix} \lambda - B_{1,1} & B_{1,2} \\ B_{2,1} & \lambda - B_{2,2} \end{vmatrix} =$ $(\lambda - B_{1,1})(\lambda - B_{2,2}) - B_{1,2}B_{2,1} = \lambda^2 - tr(B)\lambda + \det B = \lambda^2 - tr(B) + 1$.
By the Cayley-Hamilton theorem, $B^2 - tr(B)B + \mathcal{I}_2 = \mathcal{O}_2$.
Multiply by AB^{-1} to obtain $AB - tr(B)A + AB^{-1} = \mathcal{O}_2$.
Taking the trace of both sides gives the desired result.