Competitive Programming and Mathematics Society

Combinatorics

Workshop 2, Week 5, Term 2, 2021

CPMSoc Mathematics

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What is combinatorics?

Combinatorics involves questions about the size of sets.

For example, the set of all ordered triples of numbers from X has size $|\{(a,b,c) : a,b,c \in X\}| = |X^3| = |X|^3 = 3^3 = 27$.

Factorisation

The *Cartesian product* of two sets is defined as $A \times B = \{(a, b) : a \in A, b \in B\}$.
The notation $A \times A \times \ldots \times A = A^n$ is also common.

Theorem (Fundamental Principle of Counting)

The cardinality of a Cartesian product is the product of the cardinalities.

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The *equivalence relation* $|A| = |B|$ groups sets into *equivalence classes* by their size, or *cardinality*.

Bijection

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There are now nine objects on the table: two dividers and seven slices. Choosing where to place the dividers uniquely determines the selection of slices, and knowing the selection of slices tells us where to place the dividers.

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There are now nine objects on the table: two dividers and seven slices. Choosing where to place the dividers uniquely determines the selection of slices, and knowing the selection of slices tells us where to place the dividers. This bijective correspondence gives us an answer of $\binom{9}{2} = 36$.

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Suppose we have some set N and integers k such that $0 \leq k \leq n = |N|$.

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- Pascal's rule: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

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$$\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = (x+y)^k$$

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- Hockey-stick identity:

$$\binom{n+1}{k+1} = \sum_{j=k}^{n} \binom{j}{k}$$

Inclusion-Exclusion

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Example

An absent-minded delivery driver has n pizzas to deliver to n different addresses. In how many ways can they deliver the pizza, one to each address, so that no pizza is delivered to its correct address?

We will call a pizza that gets delivered to the correct address a *fixed point*.
Let P be the set of all permutations, and P_j be the set of all permutations where pizza j gets correctly delivered.

$$\text{Permutations with no fixed points} =$$

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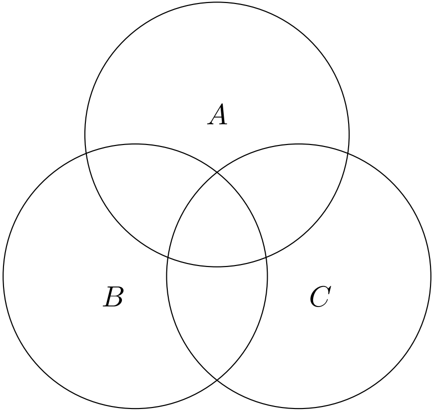
$$\text{But what is } \left| \bigcup_{j=1}^{n} P_j \right| \text{?}$$

Inclusion-Exclusion

Theorem (Principle of Inclusion-Exclusion for Two Sets)

$|A \cup B| = |A| + |B| - |A \cap B|$ *and* $|A \cap B| = |A| + |B| - |A \cup B|$.



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$$= \sum |X_j| - \sum_{i \neq j} |X_i \cap X_j| + \sum_{i \neq j, j \neq k, i \neq k} |X_i \cap X_j \cap X_k| - \ldots$$

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$$\left| \bigcup_{k=1}^{n} X_k \right| = \sum_{S \subseteq \{X_1, X_2, \ldots X_n\}} (-1)^{|S|-1} \left| \bigcap_{X_j \in S} X_j \right|$$

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Double Counting

Double-counting is short for counting the same thing in two different ways.

Example

At a maths workshop, each person knew exactly 22 others. For any pair of people X and Y who knew one another, there was no other person at the workshop whom they both knew. For any pair of people X and Y, who did not know one another, there were exactly 6 other people whom both of them knew. How many people were at the workshop?

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Define a *vee* to be a triple of people such that exactly two of the three pairs of aquaitances know each other. We count the number of vees in two different ways. Suppose there are n people at the workshop.

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Define a *vee* to be a triple of people such that exactly two of the three pairs of aquaitances know each other. We count the number of vees in two different ways. Suppose there are n people at the workshop.

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- Hence there are $231n$ vees.

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- Adding the degrees of each vertex gives $22n$ but we have overcounted the edges by a factor of 2 so there are $11n$ edges.
- Total number of edges not present is $\binom{n}{2} - 11n$.

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- Total number of edges not present is $\binom{n}{2} - 11n$.
- Every non-edge makes a vee with 6 other edges.
- Total number of vees is $6(\binom{n}{2} - 11n)$.

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- Equating expressions yields $231n = 6(\binom{n}{2} - 11n)$.
- Solving this yields $n = 100$.

Combinatorial Reciprocal Principle

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Theorem (Combinatorial Reciprocal Principle)

Let f be a function defined on a set S.
Let L_j be the set $\{x \in S : f(x) = j\}$.
Then the following identity holds:

$$\sum_{k \in S} \frac{1}{|L_{f(k)}|} = \sum_{j \in f(S)} \frac{|L_j|}{|L_j|} = |f(S)|$$

Combinatorial Reciprocal Principle

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Example

A maths olympiad has students from 13 different countries, and from 5 different age groups.
Show that at least nine students had more students in their age group than students from their country.

Let S be the set of students, A be the set of age groups and C be the set of countries. Then let $a : S \to A$ and $c : S \to C$ be functions returning a given student's age group and country respectively. Let $A_j = \{s \in S : a(s) = j\}$ and $C_j = \{s \in S : c(s) = j\}$.
By the combinatorial reciprocal principle,

$$\sum_{x \in S} \left(\frac{1}{C_{c(s)}} - \frac{1}{A_{a(s)}} \right) = \sum_{x \in S} \frac{1}{C_{c(s)}} - \sum_{x \in S} \frac{1}{A_{a(s)}}$$

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Since $A_{a(s)}, C_{c(s)}$ are positive integers, we have $\frac{1}{C_{c(s)}}-\frac{1}{A_{a(s)}}<1$.
We therefore require more than 8 students for which $\frac{1}{C_{c(s)}}-\frac{1}{A_{a(s)}}>0$

Recurrence Relations

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Example

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For example: $()()()(), (()())(), (()(()))$ are balanced, while $())()(()$ is not.

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$$(w_j)w_k,$$

where w_j is a j-bracketing, w_k is a k-bracketing and $j + k + 1 = n$.

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$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}.$$

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From here, we can see that

$$C_4 = C_0C_3 + C_1C_2 + C_1C_2 + C_0C_3$$
$$C_3 = C_0C_2 + C_1C_1 + C_2C_0$$
$$C_2 = C_0C_1 + C_1C_0$$

Observe also that $C_1 = C_0 = 1$. Hence $C_2 = 2$,

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$$C_3 = C_0C_2 + C_1C_1 + C_2C_0$$
$$C_2 = C_0C_1 + C_1C_0$$

Observe also that $C_1 = C_0 = 1$. Hence $C_2 = 2, C_3 = 5,$

Recurrence Relations

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Example

How many strings of four left and four right parentheses are balanced?
For example: $()()()(), (()())(), (()(()))$ are balanced, while $())()(()$ is not.

This gives us the recurrence relation

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}.$$

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Generating Functions

The *generating function* of a sequence $S = S_1, S_2, \ldots$ is defined as

$$G_S(x) = \sum_{n=0}^{\infty} S_n x^n.$$

Example

Find a general formula for C_n.

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$$

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Generating Functions

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Example

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$$G_C(x) - xG_C(x)^2 = 1$$

Generating Functions

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Example

Find a general formula for C_n.

$$G_C(x) - xG_C(x)^2 = 1 \implies G_C(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

Choose the $-$ sign so that the function has a power series expansion at zero. A Taylor series expansion tells us that

$$\sqrt{1+y} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{4^n(2n-1)} \binom{2n}{n} y^n$$

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$$\sqrt{1+y} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{4^n(2n-1)} \binom{2n}{n} y^n \text{ and with further computation, } G_C(x) = \sum_{x=1}^{\infty} \binom{2n}{n} \frac{x^n}{n+1}.$$