PROBLEM SET 3

1 Let n be a positive integer. The n cards of a deck are labelled $1, 2, \ldots n$. Starting with the deck in any order, repeat the following operation: if the card on top is labelled k, reverse the order of the first k cards. Prove that eventually the first card will be 1 (so no further changes occur).

2 All the chairs in a classroom are arranged in a square $n \times n$ array end every chair is occupied by a student. The teacher decides to rearrange the students according to the following two rules:

- Every student must move to a new chair.
- A student can only move to an adjacent chair in the same row or to an adjacent chair in the same column. In other words, each student can move only one chair horizontally or vertically.

(Note that the rules allow two students in adjacent chairs to exchange places.) Show that this procedure can be done if n is even, and cannot be done if n is odd.

3 Prove that if you add the reciprocals of a sequence of consecutive positive integers, the numerator of the sum (in lowest terms) will always be odd. For example, $\frac{1}{7} + \frac{1}{8} + \frac{1}{9} = \frac{191}{504}$.

4 A lock has 16 keys arranged in a 4×4 array, each key oriented either horizontally or vertically. In order to open it, all the keys must be vertically oriented. When a key is switched to another position, all the other keys in the same row and column automatically switch their positions too. Show that no matter what the starting positions are, it is always possible to open the lock.

5 An ordered triple of numbers is given. It is permitted to perform the following operation on the triple: to change two of them, say a and b, to $\frac{a+b}{\sqrt{2}}$ and $\frac{a-b}{\sqrt{2}}$. Is it possible to obtain the triple $(1, \sqrt{2}, 1+\sqrt{2})$ from the triple $(2, \sqrt{2}, \frac{1}{\sqrt{2}})$ using this operation?

6 There are 2000 white balls in a box. There are also unlimited supplies of white, green and red balls, initially outside the box. During each turn, we can replace two balls in the box with one or two balls as follows: two whites with a green, two reds with a green, two greens with a white and red, a white and a green with a red, or a green and red with a white.

(a) After finitely many of the above operations there are three balls left in the box. Prove that at least one of them is green.

(b) Is it possible that after finitely many operations only one ball is left in the box?