

Competitive Programming and Mathematics Society

# Number Theory Workshop 2, Week 6, Term 1, 2021

**CPMSoc Mathematics** 

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- Key techniques include modular arithmetic and algebraic manipulations.





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#### It is transitive:

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3 \equiv 7 \pmod{4} and 7 \equiv 15 \pmod{4} implies 3 \equiv 15 \pmod{4}, just as a = b, b = c \implies a = c.
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- It is symmetric:

 $a \equiv b \pmod{n}$  implies  $b \equiv a \pmod{n}$ , just as  $a = b \implies b = a$ .

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- If  $a \equiv b \pmod{n}$ , then  $a^k \equiv b^k \pmod{n}$ .
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- If p is prime, then  $a^p \equiv a \pmod{p}$  (Fermat's little theorem).
- p is prime iff  $(p-1)! \equiv -1 \pmod{p}$  (Wilson's theorem).



#### Example



#### Example

Can you find a set of 2000 distinct positive integers such that the sum of the members of every subset is not a perfect square?

**1** Consider the set of integers given by  $n_i = 3 \cdot 4^i$  for i = 1, 2, ..., 2000.



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- 7 Thus, every subset sum is not a perfect square.

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4 | 8 implies 8 /4, just as x \ge y implies y \not\ge x unless y = x.
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- It is antisymmetric:
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  - If  $a \mid b$ , then  $b \not| a$  unless b = a.
- Divisibility is not a total order, since  $4 \not| 7$  and  $7 \not| 4$ , while at least one of  $x \ge y$  or  $y \ge x$  must always be true.

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#### Example

Show that for any non-negative integer n,  $1^n + 2^n + 3^n + 4^n$  is divisible by 5 if and only if n is not divisible by 4.

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**1** If 4|n, then n = 4k. Note that  $1^4 \equiv 1 \pmod{5}$ ,  $2^4 = 16 \equiv 1 \pmod{5}$ ,  $3^4 = 81 \equiv 1 \pmod{5}$ ,  $4^4 = 256 \equiv 1 \pmod{5}$ .

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 Otherwise when n ≡ 1 (mod 4): 1<sup>4k+1</sup> + 2<sup>4k+1</sup> + 3<sup>4k+1</sup> + 4<sup>4k+1</sup> ≡ 1 + 2 + 3 + 4 ≡ 10 ≡ 0 (mod 5)

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 When n ≡ 2 (mod 4): 1<sup>4k+2</sup> + 2<sup>4k+2</sup> + 3<sup>4k+2</sup> + 4<sup>4k+2</sup> ≡ 1 + 4 + 9 + 16 ≡ 30 ≡ 0 (mod 5)

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 When n ≡ 3 (mod 4): 1<sup>4k+3</sup> + 2<sup>4k+3</sup> + 3<sup>4k+3</sup> + 4<sup>4k+3</sup> ≡ 1 + 8 + 27 + 64 ≡ 100 ≡ 0 (mod 5)

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- 6 Therefore, true! (by cases)





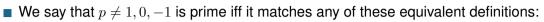


# We say that p ≠ 1,0,-1 is prime iff it matches any of these equivalent definitions: 1 Whenever p | ab, p | a or p | b.



- We say that  $p \neq 1, 0, -1$  is prime iff it matches any of these equivalent definitions:
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  - 2 Whenever p = ab,  $a = \pm 1$  or  $b = \pm 1$ .





1 Whenever  $p \mid ab, p \mid a \text{ or } p \mid b$ .

2 Whenever 
$$p = ab$$
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#### Theorem

Chebyshev said, and I'll say it again - there is always a prime between n and 2n.





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■ We say that *a* and *b* are coprime or relatively prime iff:





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#### Theorem

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We say that a and b are coprime or relatively prime iff:
Whenever c | a and c | b we must have c = 1, 0, -1.





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#### Theorem

Chebyshev said, and I'll say it again - there is always a prime between n and 2n.

■ We say that *a* and *b* are coprime or relatively prime iff:

```
1 Whenever c \mid a and c \mid b we must have c = 1, 0, -1.
```

**2** GCD(a,b) = 1. (sometimes written as (a,b) = 1)



#### Example



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Let n be a positive integer such that  $2^n - 1$  is a prime number. Prove that n is a prime number.

# **1** Suppose *n* is not prime. Let n = xy where $x, y \in \mathbb{N}$ and $x, y \ge 2$ . Then $2^n - 1 = 2^{xy} - 1$ .



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- 5 Thus, *n* must be prime!



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Prove that there are infinitely many distinct pairs (a, b) of relatively prime integers a > 1and b > 1 such that  $a^b + b^a$  is divisible by a + b.



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Since gcd(2k + 1, 2k - 1) = 1, and we can take any k > 1, we have infinitely many distinct pairs satisfying the conditions.