PROBLEM SET 2

1 Prove or disprove: For all pairs of positive integers (a, b), there exists some positive integer n such that an is a perfect cube, while bn is a perfect fifth power.

2 Determine all positive integers relatively prime to the terms of the finite sequence $a_n = 2^n + 3^n + 6^n - 1$, where $n \geq 1$.

3 A deck of $n \geq 1$ cards is given. A positive integer is written on each card. The deck has the property that arithmetic mean of the numbers on each pair of cards is also the geometric mean of the numbers on some collection of one or more cards. For what n does it follow that the numbers on all the cards are equal?

4 An integer sequence is defined by $a_n = 2a_{n-1} + a_{n-2}$, $a_0 = 0$, $a_1 = 1$. Prove that 2^k divides a_n if and only if 2^k divides n.

5 Find all pairs of integers (a, b) for which there exist functions $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ satisfying $f(g(x)) = x + a$ and $g(f(x)) = x + b$ for all integers x.

6 Find all pairs of positive integers $m, n \geq 3$ for which there exist infinitely many positive integers a such that $\frac{a^m+a-1}{a^n+a^2-1}$ is itself an integer.