1 Problem Set 1 Solutions

A1 Prove that $\log_{10} 2$ is irrational.

Solution

Assume for the sake of contradiction that $\log_{10} 2$ is rational. Since it is positive, then by assumption we can write $\log_{10} 2 = \frac{p}{q}$ for some positive integers p and q such that $\gcd(p, q) = 1$. Taking 10 to the power of both sides gives $2 = 10^{\frac{p}{q}}$, so $2^q = 10^p$. Since $p \geq 1$, then 10^p is divisible by 5 as $5|10$, but 5 does not divide 2^q, as it is a power of 2, which yields a contradiction. Hence, $\log_{10} 2$ is irrational.

A2 Show that there does not exist a function $f : \mathbb{Z} \to \{1, 2, 3\}$ such that $f(x) \neq f(y)$ for all $x, y \in \mathbb{Z}$ such that $|x - y| \in \{2, 3, 5\}$.

Solution

Assume that such an f exists. We focus on some particular function values. Let $f(0) = a$ and $f(5) = b$, where $a, b \in 1, 2, 3, a \neq b$. Since $|5 - 2| = 3$, $|2 - 0| = 2$, we have $f(2) \neq a, b$ so $f(2)$ is the remaining number, say c. Finally, because $|3 - 0| = 3$, $|3 - 5| = 2$, we must have $f(3) = c$. Therefore, $f(2) = f(3)$. Translating the argument to an arbitrary number x instead of 0, we obtain $f(x + 2) = f(x + 3)$ and so f is constant. But this violates the condition from the definition. It follows that such a function does not exist.

A3 Every point of three-dimensional space is coloured red, green or blue. Show that one of the colours attains all distances; that is, any positive real number represents the distance between two points of this colour.

Solution

Arguing by contradiction, we assume that none of the colours has the desired property. Then there exist distances $r \geq g \geq b$ such that r is not attained by red points, g by green points, and b by blue points (without loss of generality).

Consider a sphere of radius r centred at a red point. Its surface has green and blue points only. Since $g, b \leq r$, the surface of the sphere must contain both green and blue points. Choose a green point M on the sphere. There exist two points P and Q on the sphere such that $MP = MQ = g$ and $PQ = b$, since $g \geq b$. So on the one hand, either P or Q is green, or else P and Q are both blue. Then either there exist two green points at distance g, namely M and P, or M and Q, or there exist two blue points at distance b. This contradicts the initial assumption. The conclusion follows.

A4 Show that no set of nine consecutive integers can be partitioned into two sets with the product of the elements of the first set equal to the product of the elements of the second set.

Solution

Assume that such numbers do exist, and let us look at their prime factorisations. For primes p greater than 7, at most one of the numbers can be divisible by p, and the partition cannot exist. Thus the prime factors of the given numbers can only be $2, 3, 5$ and 7.

We now look at repeated prime factors. Because the difference between two numbers divisible by 4 is at least 4, at most three of the nine numbers are divisible by 4. Also, at most one is divisible by 9, at most one by 25, and at most one by 49. Eliminating there at most $3 + 1 + 1 + 1 = 6$ numbers, we are left with at least three numbers among the nine that do not contain repeated prime factors. They are among the divisors of $2 \cdot 3 \cdot 5 \cdot 7$, and so among the numbers

$$2, 3, 5, 6, 7, 10, 14, 15, 21, 30, 35, 42, 70, 105, 210.$$

Because the difference between the largest and the smallest of there three numbers is at most 9, none of them can be greater than 21. We have to look at the sequence $1, 2, 3, \ldots 29$. Any subsequence of consecutive integers of length 9 that has a term greater than 10 contains a prime number greater than or equal to 11, which is impossible. And from $1, 2 \ldots 10$, we cannot select nine consecutive numbers with the required property. This contradicts our assumption, and the problem is solved.

A5 Prove that infinitely many primes are one more than a multiple of 4.

Solution

Suppose $n > 1$ is an integer. We define $N = (n!)^2 + 1$. Suppose p is the smallest prime divisor of N. Since N is odd, p cannot be equal to 2. It is clear that p is bigger than n (otherwise $p|1$). If we show that p is of the form $4k + 1$ then we can repeat the procedure replacing n with p and we produce an infinite sequence of primes of the form $4k + 1$.

We know that p has the form $4k + 1$ or $4k + 3$. Since $p|N$ we have

$$(n!)^2 \equiv -1 \pmod{p}.$$

Therefore,

$$(n!)^{p-1} \equiv (-1)^{\frac{p-1}{2}} \pmod{p}.$$

Using Fermat's Little Theorem we get

$$(-1)^{\frac{p-1}{2}} \equiv 1 \pmod{p}.$$

If p was of the form $4k + 3$ then $\frac{p-1}{2} = 2k + 1$ is odd and therefore we obtain $-1 \equiv 1 \pmod{p}$ or $p|2$ which is a contradiction since p is odd. Hence, p must be of the form $4k + 1$. By making n arbitrarily large, we can find infinitely many primes that are one more than a multiple of four.

A6 Given any sequence of $mn + 1$ real numbers, show that some subsequence of length $(m + 1)$ is increasing or some subsequence of length $(n + 1)$ is decreasing.

Solution

Assume that the result is false. For each number x in the sequence, form the ordered pair (i, j) where i is the length of the longest increasing subsequence beginning with x and j is the length of the longest decreasing subsequence ending with x. Since the result is false, then $1 \le i \le m$ and $1 \le j \le n$. Thus we have $mn + 1$ ordered pairs, of which at most mn are distinct. Hence, two members of the sequence, say a and b, are associated with the same ordered pair (s, t). Without loss of generality we may assume that a precedes b in the sequence.

If $a < b$, then a, together with the longest increasing subsequence beginning with b, is an increasing subsequence of length $(s+1)$, contradicting the fact that s is the length of the longest increasing subsequence beginning with a. Hence, $a \ge b$. But then, b, together with the longest decreasing subsequence ending with a, is a s subsequence of length $(t + 1)$, contradicting that the longest decreasing subsequence ending with b is of length t. Therefore, we have reached a contradiction, so the result is true.

A7 Show that there does not exist a strictly increasing function $f : \mathbb{N} \to \mathbb{N}$ satisfying $f(2) = 3$ and $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{N}$

Solution

Arguing by contradiction, let us assume that such a function exists. Set $f(3) = k$. Using the inequality $2^3 < 3^2$, we obtain

$$3^3 = f(2)^3 = f(2^3) < f(3^2) = f(3)^2 = k^2,$$

hence $k \geq 6$. Similarly, using $3^3 < 2^5$, we obtain

$$k^3 = f(3)^3 = f(3^3) < f(2^5) < f(2)^5 = 3^5 = 243 < 343 = 7^3.$$

This implies that $k \leq 6$, and consequently k can only be equal to 6. Thus, we should have $f(2) = 3$ and $f(3) = 6$.The monotonicity of f implies that $2^u < 3^v$ if and only if $3^u < 6^v$, where u, v are positive integers. Taking logarithms this means that $\frac{u}{v} < \log_2 3$ if an only if $\frac{u}{v} < \log_3 6$. Since the rationals are dense in the reals, it follows that $\log_2 3 = \log_3 6$. This can be written as $\log_2 3 = \frac{1}{\log_2 3} + 1$, and so $\log_2 3$ is the positive solution of the quadratic equation $x^2 - x - 1 = 0$, which is the golden ratio $\frac{1+\sqrt{5}}{2}$. The equality

$$2^{\frac{1+\sqrt{5}}{2}} = 3$$

translates to $2^{1+\sqrt{5}} = 9$. But this would imply

$$65536 = 2^{5\times 3.2} < 2^{5(1+\sqrt{5})} = 9^5 = 59049.$$

We have reached a contradiction, which proves the the function f cannot exist.