A) Let A be an $m \times m$ matrix with integer entries. Prove that if A^n converges as $n \to \infty$ for integral n, it will converge to A^m.

Note: we define the limit of A^n as the limit applied to each of A^n's entries.

B) A tree (connected undirected graph with no cycles) is grown one vertex at a time. At each step, a vertex is uniformly selected at random, and a new vertex is added along with an edge connecting it and the selected vertex. Starting from a single vertex, as the size of the tree grows, what is the limit of the expected proportion of leaf nodes in the tree?

C) Let $\mathcal{S}$ be a set of positive integers larger than 1. Suppose that for every positive integer n, there exists $s \in \mathcal{S}$ such that $\gcd(s, n) = 1$ or $\gcd(s, n) = s$. Show that there exist two integers (not necessarily distinct) $s, t \in \mathcal{S}$ such that $\gcd(s, t)$ is prime.

D) Alice and Bob are trying to communicate information to each other. Let n, k be positive integers with $n \geq k > 1$. Alice will randomly receive k distinct positive integers less than or equal to n, and will choose to send $k - 1$ of them to Bob, one by one. Using only the integers he receives in sequence, Bob must figure out what the number Alice didn't send was. If Alice and Bob agree upon a strategy beforehand, and Bob can always guess the missing number without any extra information from Alice, what is the maximum value of n in terms of k?

E) *The following problem is open in the sense that the answer to part (b) is not currently known. A solution to part (a) will be awarded 7 points. Up to 7 additional points may be awarded for progress in part (b).*
Let n, d be positive integers. Let $\mathcal{A}$ be the set of all integer points on the x-y plane where both x and y components are non-negative integers less than n.
We define $K(p, q)$, for integers p, q, to be the smallest non-negative integer k such that $p + k \equiv q \pmod{n}$ or $p - k \equiv q \pmod{n}$. Furthermore, define $D((g, h), (i, j)) = K(g, i) + K(h, j)$ for all $(g, h), (i, j) \in \mathcal{A}$.

A subset $\mathcal{S}$ of $\mathcal{A}$ is chosen such that for all pairs of points $(g, h), (i, j) \in \mathcal{S}$, $D((g, h), (i, j)) \geq 2d$. With proof, find the maximum size of S (in terms of n and d):

a) when n is divisible by $2d$,
b) in all other cases.