Competitive Programming and Mathematics Society

Launch Week contest debrief

CPMSoc

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Attendance form :D

Summary

Find

$$2^1 - 2^2 + 2^3 - \cdots - 2^{10}.$$

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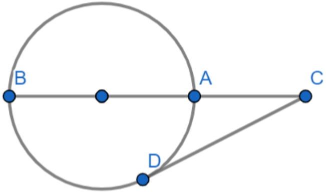
Negative of geometric series with $r = -2$, $a = 2$.

Frankly A Miracle



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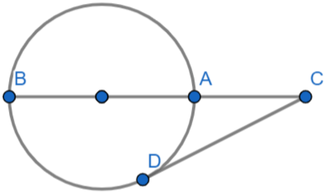

Two options:

- Directly apply tangent secant theorem ($CD^2 = AC \cdot BC$)
- Construct a line from circle center to D, note it's perpendicular to CD and use Pythagoras

The latter gives $AC = \sqrt{\frac{1}{4}AB^2 + CD^2} - \frac{1}{2}AB$ (note $CD = AB$), with $AB = 2$.

Geoff's Leftovers

Geoff the Geometer has a certain number of identical side-length-1 squares which can all be tiled to form larger shapes. However, if Geoff tries to form squares with side lengths 2, 3, 4, 5 or 6, he will always have 1 square left over. We know that Geoff has at least 3,000 and at most 10,000 squares. What is the sum of the possible numbers of squares that Geoff has?

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- This number must leave a remainder 1 when divided by $4, 9, 16, 25, 36$.
- Thus it leaves a remainder of 1 when divided by their LCM, which is 3600.
- This gives us two numbers in the range $[3,000, 10,000]$: 3601 and 7201, thus our answer is $3601 + 7201 = 10802$.

Probably

Abby, Ben, Cyril and David are one by one randomly and independently assigned positive integers, where the probability of being assigned x is

$$P(X = x) = \frac{6}{\pi^2 x^2}.$$

What is the probability that their numbers are in increasing order (with respect to the ordering above)?

Mistake in question: should be distinct numbers (original question involved a continuous probability distribution). Every permutation of 4 sets of numbers is equally likely to occur, and exactly one involves them being increasing. So, the probability is $\frac{1}{4!} = \frac{1}{24}$

Knights and Knaves

100 knights and knaves stand around in a circle, and each of them give a statement about the two people besides them under the following conditions:

- Knights always tell the truth.
- Knaves always lie.

For each scenario, how many different possible number of knights are there?

1. Everyone says "there are two knights next to me".
2. Everyone says "there are no knights next to me".
3. Every odd person says "there is exactly one knight next to me", and every even person says "there are exactly two knights next to me"

Knights and Knaves - Answer

For the first question, the existence of a single knight means that there are two knights surrounding that knight, and two knights around those knights, and so on, propagating until everyone is a knight. Otherwise, there are no knights, giving two possible numbers.

For the second, all the people can't be a knave, so there must be a knight who is surrounded by two knaves. Following this is either a knight, followed by a knave, or a knave followed by a knight and a knave. This means all possible configurations involving stitching these length 3 and 2 sequences together (note two FTF sequences can overlap, but this is equivalent to stitching FTF with TF). The lowest number of knights is given by stitching 32 FTF sequences followed by two TF sequences, which is 34 (we cannot have 33 as 33 FTF sequences leaves one person unaccounted for, who must be alright anyway), and the highest is 50 TF sequences. $50 - 34 + 1 = 17$

Knights and Knaves - continued

For the last question, suppose there is at least one knight. If it's in an even position, it has exactly two surrounding it in odd positions. Both of those knights can only have one knight next to them, so knaves are in the remaining position. If the knight is in an odd position, one side has a knight and the other a knave; the knight is in an even position, once again giving us three knights surrounded by knaves. Therefore, knights come in isolated contiguous sequences of 3 centered around an even position. Now, the "bordering" knave (even position) must be followed by a knave (cannot have two knights), which is followed by a knave (cannot have exactly one knight), from which point a knight can follow or two more knaves. The lowest number of knights is 0 (all knaves), and the highest is 48. This is because if we try and fit as many groups of three knights, since they must be surrounded by groups of three knaves, we have 16 groups and two spots left that must be filled by knaves. Note that every other multiple of 3 can be achieved by erasing a group of three knights, erasing one of the groups of knaves bordering, and filling in with knaves (note there are an even number of spaces so this is valid). Thus we have 17 multiples of 3 from 0 to 48, which is our answer.

Cyclic

Find three values of a such that $f(x) = \frac{1}{a-x}$ is cyclic with order $5, 6, 7$ (respectively).

We'll approximate x as a rational number i.e. let $x = \frac{p}{q}$ for integers p, q. Now, note $f(x) = \frac{1}{a-\frac{p}{q}} = \frac{q}{aq-p}$. If we represent our number as a vector of the numerator and denominator, then $f(x)$ can be represented by

$$\begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}.$$

Let the prior matrix be $\mathbf{A}$. Note that vectors of equivalent fractions are scalar multiples of each other, so we can find an a that makes f cyclic with order n by analysing eigenvalues!

Cyclic - Continued

Since $\mathbf{A}$ is a real matrix, if A has an eigenvalue with principal argument $\frac{\pi}{n}$, then the other eigenvalue is a conjugate of it, and thus has argument $-\frac{\pi}{n}$ (note both eigenvalues have the same modulus, call this k). Then, if our respective eigenvectors are $\vec{v}, \vec{w}$, then since $\vec{v}, \vec{w}$ are linearly independent and thus span $\mathbb{C}^2$, we may write

$$\mathbf{A}^n\vec{u} = \mathbf{A}^n(c_1\vec{v} + c_2\vec{w}) = \mathbf{A}^n\vec{v} + \mathbf{A}^n\vec{w} = k^n\vec{v} + k^n\vec{w} = k^n\vec{u}.$$

Cyclic - Finale

The eigenvalues of A are the roots of $t(t-a)+1 = t^2 - at + 1$, which are

$$\frac{a \pm \sqrt{a^2-4}}{2}.$$

Note $\frac{a}{2}$ is real, so $\sqrt{a^2-4}$ must be imaginary. To get the eigenvalue to have the desired argument, we want $\frac{\sqrt{4-a^2}}{a} = \tan\left(\frac{\pi}{n}\right)$. Rearranging, we obtain

$$4 = a^2 + a^2\tan^2\left(\frac{\pi}{n}\right) \implies a = \pm\frac{2}{\sqrt{1+\tan^2\left(\frac{\pi}{n}\right)}},$$

which upon substituting the appropriate values of n gives us our desired values for a.

Find Those Functions

Find all

$$f(f(x)+f(y)) = f(f(f(x))) + y$$

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A lot of mistakes in people's proofs!

- Didn't prove surjectivity (e.g. claimed $f(f(x)) = f(x) + c$ for all x meant $f(x) = x + c$)
- Claimed f has an inverse without proof
- Claimed f was differentiable
- Carry-overs (most commonly bracketing e.g. $f(f(f(x)))$ becomes $f(f(x))$)

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We've covered two important techniques in our workshops which can help us solve this: proving injectivity/surjectivity, and exploiting symmetries/non-symmetries.

Find Those Functions - Our Proof

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Fix x and vary y: $f(f(x)+f(y))$ attains every real value, so f must also. (Surjectivity)

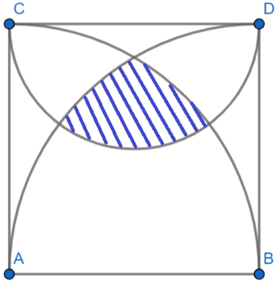
Notice swapping x and y changes only the RHS, so $f(f(f(x)))+y=f(f(f(y)))+x$. Thus, if $f(x)=f(y)$, $x=y$ (Injectivity).

Alternatively, from the symmetry argument, $f(f(f(x)))+0=x+f(f(f(0)))$ (f^3 is linear and thus injective and surjective, so f must also be).

Finally, we substitute $x \to f(x)$ (surjectivity), $y \to 0$ to obtain $f(f(x)+f(0))=f(f(x))$, giving $f(x)=x+f(0)$ (injectivity).

Substituting back into the functional equation shows the family $f(x)=x+c$, for any real c, is a complete set satisfying our equation

Eye Of CPM



Easy way: integration (and probably how you should do it). Hard way:

```
L = pi / 6 - sqrt(3) / 4
top = 1 - pi / 4 - (pi / 12 - L)
theta = atan(1/2)
lensupper = pi * 2*theta / 2/pi - 1/2 * sin(2*theta)
lenslower = pi / 4 * (pi - 2*theta) / 2/pi - 1/8 * sin(2*theta)
answer = (lensupper + lenslower) * 2 + top - pi / 8
```

Polyhedra

The net is a 2-dimensional shape formed by opening up the surface of a polyhedron so that it can be laid flat on a plane. This necessarily requires breaking the surface along a number of edges. Show that to create a connected net of an arbitrary convex polyhedron, the number of edges that must be broken is exactly one less than the number of its vertices.

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- Consider the subgraph of the face graph formed by which faces are connected by the net, and the subgraph of the vertex graph formed by which edges have been cut to form the net.
- Note that due to the topology of the graph, if one is acyclic, the other is connected.
- The question specifies that the net is connected therefore the faces graph must be connected.

Polyhedra

- Consider the situation that the face graph has a cycle
- That is, there exists a loop of connected coplanar faces that used to be in the original polyhedron
- This is impossible if the original polyhedron was strictly convex.
- Together, this means the face graph forms a spanning tree of the original polyhedron, and hence the number of edges in the net is $F - 1$ where F is the number of faces.
- As the number of broken edges is the total number of edges minus the edges that remained intact, we get $E - F + 1$ broken edges.
- Due to the polyhedron being convex, it has an Euler characteristic of 2 and hence $F = E + 2 - V$ therefore the number of broken edges is $V - 1$.

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n real random numbers are selected uniformly from the range $[1, 2]$ and their geometric mean is calculated. What is the limit of the expected value of this mean as $n \to \infty$?

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$$\begin{aligned}\lim_{n\to 0} 2^{n+1} - 1 &= \lim_{n\to 0} 2e^{n\ln 2} - 1 \\ &= \lim_{n\to 0} 2(n\ln 2 + 1) - 1\end{aligned}$$

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- Using the converse of our aforementioned identity, we can deduce that $\lim_{n\to 0} 2n\ln 2 + 1 = e^{2n\ln 2} = 4^n$ and thus our numerator becomes 4. This gives us an answer of $\frac{4}{e}$.

Cards

A game is played with a single dealer and a single player. The dealer will place some cards in a line on a table, from left to right, each with a different number on it. The player will point to a card and pick a direction. The dealer will then flip the card that was pointed to and, for each subsequent card in the player's chosen direction, flip it if and only if the card has a higher number than the one they flipped last, until they reach an end. The player's score is the number of cards that were flipped over, and the dealer will order the cards in a way that minimises the player's maximum possible score. What is the maximum number of cards the dealer can deal to guarantee the player doesn't score more than 10?

Cards - Exploring

For a score of 2, we can fit at most 5 cards $(2, 1, 5, 3, 4)$, but 6 cards will always give a score of 3.

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To decide this, imagine placing the highest card first. Note that all cards to the left are "independent" to those of the right (any increasing sequence on the right will either go to the right, or stop at the 10, and similarly for the left).

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Now we maximise how many we can place on the left side by first placing the maximum, and repeat.

Cards - Our proof

Let $f(m, n)$ denote the maximum number of cards that can be packed with increasing sequences of length m going to the left and n going to the right. Note we can write

$$f(m, n) = f(m-1, n) + f(m, n-1) + 1.$$

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$$f(m, n) = f(m-1, n) + f(m, n-1) + 1.$$

Since $m, n > 0$, we draw a triangle representing this recurrence relation, giving us Pascal's triangle but each number one less. Specifically, $f(n, n) = \binom{2n}{n} - 1$.

Trivial Polynomials

Find all complex polynomials P and real constants k such that

$$P(2x)P(x+k) \mid P(x^2).$$

Note $\deg(P(2x)) = \deg(P(x+k)) = P(x)$, so

$$\deg(P(2x)P(x+k)) = \deg(P(2x)) + \deg(P(x+k)) = 2\deg(P(x)) = \deg(P(x^2)).$$

Thus divisiblility is achieved if and only if both polynomials have exactly the same set of roots with the exact same multiplicities (equality is achieved when this condition is met and the leading coefficients are equal, since this implies both have the same factorisation, but here we don't care about the latter).

Trivial Polynomials - Our proof

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Firstly, note the trivial case is if P has no roots: $P(x) = c$ for any complex c. Then, any real k will satisfy our equation. Let's now ignore this case. If z is a root of $P(x^2)$, then z^2 is a root of $P(x)$, so $\frac{z^2}{4}$ must then be a root of $P(2x)$, which is, coming full circle, a root of $P(x^2)$. Thus, z being a root of $P(x^2)$ implies $\frac{z^2}{4}$ is a root of $P(2x)$. From here, we can construct an infinite sequence of roots by picking $z_0 = z$ (for some root z we choose) and ruling $z_n = \frac{z_{n-1}^2}{4}$. Note that $\{|z_n|\}_n$ is a strictly decreasing sequence if $0 < |z| < 4$ and a strictly increasing sequence if $\{|z| > 4\}$. Thus, since P cannot have infinite roots as a polynomial, $|z| = 4$ or $|z| = 0$.

Trivial Polynomials - Continued

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Now, what does this look like on the complex plane? Well the roots of $P(x)$ lie on the origin or on the circle with radius 4, so the roots of $P(x^2)$ lie on the origin or on the circle with radius $\sqrt{4} = 2$. For simplicity assume 0 has principal argument 0, and suppose there exist a non-empty sequence of roots $x_1, \ldots, x_t$ with a non-zero principal argument. So, $x_1 - k, \ldots, x_t - k$ are all roots of $P(x + k)$ and thus also $P(x^2)$, so they lie on the circle with radius 2. Note, however, we see that because k corresponds to a horizontal translation and $\sqrt{16 - x^2} - \sqrt{4 - x^2}$ has at most two solutions for any given y-value, we can only have at most 2 roots which are conjugates of each other in order for a fixed real offset to send them between circles with different radii.

Trivial Polynomials - Finale

Note $\frac{x_1}{2}, \ldots, \frac{x_t}{2}$ are all roots of $P(2x)$, and so lie on the circle of radius 2. If $k = 0$, no root can be sent between the circles, so here $P(x) = x^n$. Otherwise, since $\frac{x_i}{2}$ and $x_i - k$ have distinct principal arguments, $\frac{x_i}{2}$ is either a root with zero principal argument OR equals $\bar{x}_i - k$. Note the latter cannot happen unless x_i is a negative real number as otherwise each of their principal arguments would sit in ($(0, \pi)$ and $(\pi, 2\pi)$) respectively and thus not be equal. We find all roots of P either have 0 principal argument (non-negative real number), or is a negative real number; the only candidates for this are $-4, 0, 4$.

Trivial Polynomials - Finale

Note $\frac{x_1}{2}, \ldots, \frac{x_t}{2}$ are all roots of $P(2x)$, and so lie on the circle of radius 2. If $k = 0$, no root can be sent between the circles, so here $P(x) = x^n$. Otherwise, since $\frac{x_i}{2}$ and $x_i - k$ have distinct principal arguments, $\frac{x_i}{2}$ is either a root with zero principal argument OR equals $\bar{x}_i - k$. Note the latter cannot happen unless x_i is a negative real number as otherwise each of their principal arguments would sit in ($(0, \pi)$ and $(\pi, 2\pi)$) respectively and thus not be equal. We find all roots of P either have 0 principal argument (non-negative real number), or is a negative real number; the only candidates for this are $-4, 0, 4$.

If P has a root of -4, then $P(x^2)$ has non-real roots, but $P(2x)P(x+k)$ clearly don't; by contradiction, P only has roots 4 and 0. Substituting, we have

$$(2x-4)^n(2x)^m(x-4+k)^n(x+k)^m = 2^{n+m}(x-2)^n x^m (x-4+k)^n (x+k)^m,$$

and

$$(x^2-4)^n x^{2m} = (x+2)^n (x-2)^n x^{2m},$$

so clearly $k = 6$.

Feedback form :D