Round 1

$$\int(3x^2-4x+5)\mathrm{d}x = x^3-2x^2+5x+C$$

$$\int \tan x(\sin 2x+\cos x)\mathrm{d}x = \int(2\sin^2 x+\sin x)\mathrm{d}x = \int(1-\cos 2x+\sin x)\mathrm{d}x = x-\frac{1}{2}\sin 2x-\cos x+C$$

$$\int \frac{1}{\sqrt{x^{-2}-x^2}}\mathrm{d}x = \frac{1}{2}\int \frac{2}{x}\sqrt{1-(x^2)^2}\mathrm{d}x = \frac{1}{2}\arcsin(x^2)+C$$

$$I=\int\sqrt{1-x^2}\mathrm{d}x = x\sqrt{1-x^2}+\int\frac{x^2-1+1}{\sqrt{1-x^2}}\mathrm{d}x = x\sqrt{1-x^2}-I+\int\frac{1}{\sqrt{1-x^2}}\mathrm{d}x$$
$$\implies I=\frac{1}{2}(x\sqrt{1-x^2}+\sin^{-1}(x))+C$$

$$\int \sec x\mathrm{dx} = \int\frac{\sec x(\tan x+\sec x)}{\sec x+\tan x}\mathrm{d}x = \int\frac{\sec x\tan x+\sec^2 x}{\sec x+\tan x}\mathrm{d}x = \ln|\sec x+\tan x|+C$$

$$\int_0^{\frac{1}{2}}\frac{x^2+3}{x^3-6x^2+11x-6}\mathrm{d}x = \int_0^{\frac{1}{2}}\frac{x^2+3}{x^3-6x^2+11x-6}\mathrm{d}x = \int_0^{\frac{1}{2}}\left(\frac{2}{x-1}+\frac{-7}{x-2}+\frac{6}{x-3}\right)\mathrm{d}x$$
$$=2\ln\frac{1}{2}-2\ln 0-7\ln\frac{3}{2}+7\ln 2+6\ln\frac{5}{2}-6\ln 3 = 6\ln 5-13\ln 3+6\ln 2$$

$$\int\frac{1}{\sqrt{x}(x+1)}\mathrm{d}x = 2\int\frac{1}{2\sqrt{x}}\frac{1}{\sqrt{x}^2+1}\mathrm{d}x = 2\tan^{-1}(\sqrt{x})$$

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{x^2-1}{(x^2+1)^2}\mathrm{d}x &= 2\int_0^{\infty}\frac{1-\frac{1}{x^2}}{(x+\frac{1}{x})^2}\mathrm{d}x\\ &= 2\int_0^{\infty}\frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(x+\frac{1}{x}\right)}{(x+\frac{1}{x})^2}\mathrm{d}x\\ &= -2\left(\lim_{x\to\infty}\frac{1}{x+\frac{1}{x}}-\lim_{x\to 0}\frac{1}{x+\frac{1}{x}}\right)\\ &= -2(0-0)\\ &= 0 \qquad \text{(trig sub also works and may be easier to see)}\end{aligned}$$

$$\int 10^x \mathrm{d}x = \frac{1}{\ln(10)} \int \ln(10) e^{\ln(10)x} \mathrm{d}x = \frac{10^x}{\ln(10)} + C$$

$$\int \frac{1}{e^x + e^{-x}} \mathrm{d}x = \int \frac{e^x}{(e^x)^2 + 1} \mathrm{d}x = \tan^{-1}(e^{2x} + 1) + C$$

Round 2

$$\int (\sin(x)\cos(x) + 2023)\mathrm{dx} = \frac{1}{2} \int \sin(2x)\mathrm{d}x \ + 2023x = -\frac{1}{4}\cos(2x) + 2023x + C$$

$$\begin{aligned} \int \frac{2\cos^2 x}{\cos x + 1} \mathrm{d}x &= \int \left(\frac{2\cos^2 x + 2\cos x}{\cos x + 1} - \frac{2\cos x + 2}{\cos x + 1} + \frac{2}{\cos x + 1} \right) \mathrm{d}x \\ &= \int \left(\cos x - 1 + \sec^2 \frac{x}{2} \right) \mathrm{d}x = 2\sin x - 2x + 2\tan\frac{x}{2} + C \end{aligned}$$

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{\sin^4(x)}{\sin^4(x) + \cos^4(x)} \mathrm{d}x &= \frac{1}{2} \left(\int_0^{\frac{\pi}{2}} \frac{\sin^4(x)}{\sin^4(x) + \cos^4(x)} \mathrm{d}x + \int_0^{\frac{\pi}{2}} \frac{\sin^4(\frac{\pi}{2} - x)}{\sin^4(\frac{\pi}{2} - x) + \cos^4(\frac{\pi}{2} - x)} \mathrm{d}x \right) \\ &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{\sin^4(x) + \cos^4(x)}{\sin^4(x) + \cos^4(x)} \mathrm{d}x = \frac{1}{2} \int_0^{\frac{\pi}{2}} 1 \mathrm{d}x = \frac{\pi}{4} \end{aligned}$$

$$\int_{-\pi}^{\pi} x \sin^2(x) \cos^5(x) \mathrm{d}x = 0 \qquad (x \text{ is odd, but } \sin^2(x) \text{ and } \cos^5(x) \text{ are even, so function is odd})$$

$$\int \frac{1}{x\sqrt{1 - (\ln x)^2}} \mathrm{d}x = \sin^{-1}(\ln x) + C \qquad \text{(reverse chain rule)}$$

$\int_0^3 \frac{x^3 + 3}{x^2 - 1} \mathrm{d}x$ is undefined since we are integrating over $x = 1$, which the function does not exist over

$$\int (3x^2 - 1)\ln(x + 1)\mathrm{d}x = (x^3 - x)\ln(1 + x) - \int \frac{x(x^2 - 1)}{x + 1} \mathrm{d}x = (x^3 - x)\ln(1 + x) - \frac{x^3}{3} + \frac{x^2}{2} + C$$

$$\int \frac{x^2-1}{x^2+1}\frac{1}{\sqrt{1+x^4}}\mathrm{d}x = \int \frac{1-\frac{1}{x^2}}{x+\frac{1}{x}}\frac{1}{\sqrt{\frac{1}{x^2}+x^2}}\mathrm{d}x = \int \frac{1}{u\sqrt{(u^2-2)}}\mathrm{d}u \text{ (let } u = x+\frac{1}{x})$$

$$= \int \frac{\sqrt{u^2-2}}{u}\cdot\frac{1}{u\sqrt{u^2-2}}\mathrm{d}k \text{ (let } k=\sqrt{u^2-2})$$

$$= \int \frac{1}{k^2+(\sqrt{2})^2}\mathrm{d}k = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\sqrt{\left(x+\frac{1}{x}\right)^2-2}\right)+C$$

$$\int \frac{1}{1+x^4}\mathrm{d}x = \int \frac{1}{(x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)}\mathrm{d}x = \frac{1}{2}\int\left(\frac{-\frac{1}{\sqrt{2}}x+1}{(x-\frac{1}{\sqrt{2}})^2+\frac{1}{2}}+\frac{\frac{1}{\sqrt{2}}x+1}{(x+\frac{1}{\sqrt{2}})^2+\frac{1}{2}}\right)\mathrm{d}x$$

$$= \frac{1}{2}\int\left(\frac{-\frac{1}{\sqrt{2}}x+\frac{1}{2}}{(x-\frac{1}{\sqrt{2}})^2+\frac{1}{2}}+\frac{\frac{1}{\sqrt{2}}x+\frac{1}{2}}{(x+\frac{1}{\sqrt{2}})^2+\frac{1}{2}}+\frac{\frac{1}{2}}{(x-\frac{1}{\sqrt{2}})^2+\frac{1}{2}}+\frac{\frac{1}{2}}{(x-\frac{1}{\sqrt{2}})^2+\frac{1}{2}}\right)\mathrm{d}x$$

$$= \frac{1}{4\sqrt{2}}\ln\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right)+\frac{1}{2\sqrt{2}}\left(\tan^{-1}(\sqrt{2}x-1)+\tan^{-1}(\sqrt{2}x+1)\right)+C$$

$$\int \frac{\sin(x)}{\cos(x+\frac{\pi}{3})}\mathrm{d}x = \int \frac{\sin(x+\frac{\pi}{3}-\frac{\pi}{3})}{\cos(x+\frac{\pi}{3})}\mathrm{d}x = \int \frac{\sin(x+\frac{\pi}{3})\cos\frac{\pi}{3}-\cos(x+\frac{\pi}{3})\sin\frac{\pi}{3}}{\cos(x+\frac{\pi}{3})}\mathrm{d}x$$

$$= -\frac{1}{2}\ln|\cos\left(x+\frac{\pi}{3}\right)|-\frac{\sqrt{3}}{2}x+C$$

Semi-finals round 1

$$\int_2^\infty \ln\left(1-\frac{1}{\lfloor x\rfloor^2}\right)\mathrm{d}x = \sum_{n=2}^{\infty}\ln\left(1-\frac{1}{x^2}\right)$$

$$= \ln\left(\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)\right)$$

$$= -\ln 2 \text{ (telescoping)}$$

$$
\begin{aligned}
\int \cot^5 x \mathrm{d}x &= \int \frac{\cos^5 x}{\sin^5 x}\mathrm{d}x \\
&= \int \frac{(1-\sin^2)^2}{\sin^5 x}\cos x \mathrm{d}x \qquad \text{let } u = \sin x \\
&= \int \frac{(1-u^2)^2}{u^5}\mathrm{du} \\
&= \int \frac{1-2u^2+u^4}{u^5}\mathrm{du} \\
&= \int u^{-5} - 2u^{-3} + u^{-1}\mathrm{du} \\
&= -\frac{1}{4}u^{-4} + u^{-2} + \ln|x| \\
&= -\frac{1}{4}\sin^{-4} x + \sin^{-2} x + \ln|\sin x| + C
\end{aligned}
$$

$$
\begin{aligned}
\int \frac{e^x+1}{e^{2x}+1}\mathrm{d}x &= \int \left(\frac{e^x}{e^{2x}+1} + \frac{1+e^{2x}}{e^{2x}+1} - \frac{e^{2x}}{e^{2x}+1}\right)\mathrm{d}x \\
&= \int \left(\frac{\mathrm{d}(e^x)}{(e^x)^2+1} + \frac{e^{2x}+1}{e^{2x}+1} - \frac{1}{2}\cdot\frac{2e^{2x}}{e^{2x}+1}\right)\mathrm{d}x \\
&= \tan^{-1}(e^x) + x - \frac{1}{2}\ln|e^{2x}+1| + C
\end{aligned}
$$

Tiebreaker:
Let (x, y) be parametrically defined as $(a\cos\theta, b\sin\theta)$ where $0 \le \theta \le \pi$. Define a function f such that $y = f(x)$ for all values of x. Find

$$\int_{-a}^{a} f(x)\mathrm{d}x$$

Solution: This integral represents the area of an ellipse, which is πab

Semi-finals round 2

$$
\begin{aligned}
\int \frac{x^3+x}{x^6-3u^4+3u^2-1}\mathrm{d}x &= \int \frac{1+\frac{1}{x^2}}{x^3-3x+\frac{3}{x}-\frac{1}{x^3}}dx \\
&= -3\left(x+\frac{1}{x}\right)^{-4} + C
\end{aligned}
$$

$$\int_0^1 \frac{1}{\left\lfloor \frac{1}{x} \right\rfloor} \mathrm{d}x = \int_1^\infty \frac{1}{\lfloor x \rfloor} \frac{1}{x^2} \mathrm{d}x \qquad \text{with sub } u = \frac{1}{x}$$

$$= \sum_{n=1}^{\infty} \int_n^{n+1} \frac{1}{n} \frac{1}{x^2} \mathrm{d}x$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{n+1}$$

$$= \frac{\pi^2}{6} - \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

$$= \frac{\pi^2}{6} - 1$$

$$\int \frac{\sqrt{x}}{1 + \sqrt[3]{x}} \mathrm{d}x = \int \frac{6u^5 \cdot u^3}{1 + u^2} \mathrm{d}u \qquad (\text{with } u = \sqrt[6]{x})$$

$$= 6 \int \frac{u^8 + u^6 - u^6 - u^4 + u^4 + u^2 - u^2 - 1 + 1}{u^2 + 1} \mathrm{d}u$$

$$= 6 \left(\frac{\sqrt[6]{x}^7}{7} - \frac{\sqrt[6]{x}^5}{5} + \frac{\sqrt{x}}{3} - \sqrt[6]{x} + \tan^{-1}(\sqrt[6]{x}) \right) + C$$

Tiebreaker:

$$\int (21x^{2023} - 420x^{69}) \ln x \mathrm{d}x = \left(\frac{21x^{2024}}{2024} - \frac{420x^{70}}{70} \right) \ln x - \frac{21x^{2024}}{2024^2} + \frac{420x^{70}}{70^2} + C$$

Finals

$$\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{1+e^{\frac{1}{x}}}\mathrm{d}x &= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x+\cos x}{1+e^{\frac{1}{x}}}\mathrm{d}x \\
&= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{1+e^{\frac{1}{x}}}+\frac{\cos x}{1+e^{-\frac{1}{x}}}\mathrm{d}x \\
&= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x(1+e^{-\frac{1}{x}})+\cos x(1+e^{\frac{1}{x}})}{(1+e^{-\frac{1}{x}})(1+e^{\frac{1}{x}})}\mathrm{d}x \\
&= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x(2+e^{-\frac{1}{x}}+e^{\frac{1}{x}})}{2+e^{-\frac{1}{x}}+e^{\frac{1}{x}}}\mathrm{d}x \\
&= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\mathrm{d}x \\
&= 1.
\end{aligned}$$

Define the function $f_1(x) = e^x$ and any subsequent function $f_n(x) = e^{f_{n-1}(x)}$ where $n \in \mathbb{N}$. Find

$$\int f_n(x)f_{n-1}(x)...f_1(x)\mathrm{d}x$$

Solution: Substitute $u = e^x \implies \mathrm{du} = e^x\mathrm{d}x$

$$\int f_{n-1}(u)f_{n-2}(u)...f_1(u)\mathrm{du}$$

Rinse and repeat. Once you reverse all the substitutions, you'd eventually get the answer of $f_n(x)+C$. (Anything along the lines of this is okay)

$$\begin{aligned}
\int \frac{\cos^{1010} x}{\sin x\sqrt{\sin^{2022} x-\cos^{2022} x}}\mathrm{d}x &= \int \frac{\sec^2 x}{\tan x\sqrt{\tan^{2022} x-1}}\mathrm{d}x \\
&= \int \frac{\mathrm{d}u}{u\sqrt{u^{2022}-1}} \qquad (\text{let } u=\tan x) \\
&= \int \frac{2\sqrt{u^{2022}-1}}{2022u^{2021}\cdot u\sqrt{u^{2022}-1}}\mathrm{d}k \qquad (\text{let } k=\sqrt{\tan^{2022} x-1}) \\
&= \frac{1}{1011}\int \frac{\mathrm{d}k}{k^2+1} = \frac{1}{1011}\tan^{-1}\left(\sqrt{\tan^{2022}(x)-1}\right)+C
\end{aligned}$$