CPMSoc Welcome Week Competition with Solutions

May 2022

1 Questions

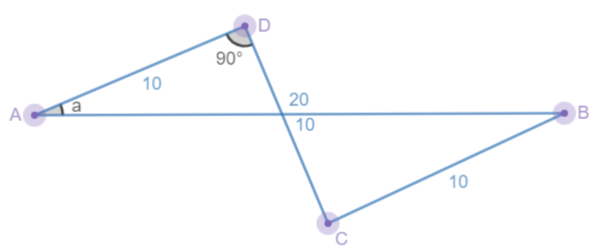
1. Find the next 2 numbers in the sequence $1, 4, 5, 2, 2, 3, 3, 0, 1, 4, \_, \_, \ldots$

   [Beginner]

   Proof. Answer: $-2, -1$.

   There's a pattern every 3 terms. □

2. In the graph below, $AD = DC = CB = 10$, $AB = 20$, $AD \perp DC$, D and C are on different sides of the line AB. Determine $\cos a$ in the form of $\frac{x+\sqrt{y}}{z}$. The answer will only accept the simplest form where x, y, z are all positive integers.

   [Beginner]

   Proof. Answer: $x = 5, y = 7, z = 8$.

   Connect AC. We can obtain $AC = 10\sqrt{2}$ with Pythagoras's Theorem. Since we now know the length of AC, CB and AB, we can apply cosine rule and get the angle for $\angle CAB$. $\angle a$ is hence $45° - \angle CAB$ and we can now calculate $\cos\angle a$. □

3. In the following list of numbers, the number n appears n times for $1 \leq n \leq 200$.

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots, 200, \ldots, 200, 200$$

What is the median of the numbers in this list?

Proof. Answer: 142. Total number of numbers in the list is $\frac{201\times200}{2} = 20100$. We are looking for the 10050th number ie. $\frac{n(n+1)}{2} \leq 10050 < \frac{(n+1)(n+2)}{2}$. Guess and check yields $n = 142$. □

4. How many sequences of zeros and ones of length 20 have all the zeros consecutive, or all the ones consecutive, or both?

(A) 190 **(B)** 192 **(C)** 211 **(D)** 380 **(E)** 382

Proof. For any a 0s, we have 20-a 1s. Consider the cases where we insert the 0 block within the 1s. There are $(20-a+1)$ ways to do this. When we insert the 1 block within the 0s, there are $(a+1)$ ways to do this. However, we have over counted the 2 cases with [all 1s][all 0s] and [all 0s][all 1s] so for any a there are $20 - a + 1 + a + 1 - 2 = 20$ ways to make such a sequence. However, for $a = 0, 20$, there's only 1 case each time that would be counted 20 times. So we let a range from 1 to 19. $19*20+2 = 382$. □

5. Find the smallest positive integer solution to $\tan 19x^\circ = \frac{\cos 96^\circ + \sin 96^\circ}{\cos 96^\circ - \sin 96^\circ}$. (Beginner)

Proof. Answer- 159 From dividing both sides by $\cos 96^\circ$ we get $\frac{1+\tan 96^\circ}{1-\tan 96^\circ} = \tan(45^\circ + 96^\circ)$, so $\tan 141^\circ = \tan 19x^\circ$, this is equivalent to solving the equation $141 + 180n = 19x$.

Take this modulo 19 and we get $9n + 8 \equiv 0 \mod 19$ or $9n \equiv 11 \equiv 144 \mod 19$. Divide both sides by 9 to get $n \equiv 16 \mod 19$. So we take $n = 16$. So then $19x = 141 + 180 \cdot 16 \Rightarrow x = 159$. This must be unique modulo 180 so the answer is 159. □

6. (2021 AMC 12A Q23) Frieda the frog begins a sequence of hops on a 3×3 grid of squares, moving 1 square up, down, left or right randomly. She does not hop diagonally. If a jump would take of off the grid, she is teleported to the opposite side. For example, if Frieda begins in the centre and makes two hops "up". the first hop places her in the top row middle square, the second hop takes her to the opposite edge, landing on the bottom row middle square. Suppose Frieda starts from the centre square, makes at most four hops at random, and stops hopping if she lands on a corner square. What is the probability that she reaches a corner square on one of the four hops?
(Beginner)

Proof. Answer $\frac{25}{32}$. We instead find the probability of not reaching any corner cells in 4 moves. By symmetry, it suffices to consider if the first move is up. The next move must be up or down. If it is up, the possibilities are UUUU, UUUD, UUUL, UUUR, UUDU, UUDD. If down, we return to the centre, and our last two moves can be any two horizontal, or any two vertical i.e $2 \times 2^2 = 8$ possible moves. This gives us a total of 14 sequences of moves not moving to any corner cells with first move being up, out of a total of $4^3 = 64$ moves. Taking the complement gives us $1 - \frac{14}{64} = \frac{25}{32}$. ☐

7. A fair 6-sided die is repeatedly rolled until an odd number appears. What is the probability that every even number appears at least once before the first occurrence of an odd number?

   (A) $\frac{1}{120}$ **(B)** $\frac{1}{32}$ **(C)** $\frac{1}{20}$ **(D)** $\frac{3}{20}$ **(E)** $\frac{1}{6}$

   Proof. We have probability $\frac{1}{2}$ of getting an even number to start out with. Then, we have a $\frac{2}{5}$ probability progressing towards our goal of 3 distinct even numbers before the first odd number. This is because getting the same even number is irrelevant, getting either of the 2 different ones progresses us, and any odd ends us. Finally, we have a $\frac{1}{4}$ probability of moving on and succeeding since there are 4 relevant numbers and 3 are odd. So, our total probability is $\frac{1}{2} \cdot \frac{2}{5} \cdot \frac{1}{4} = \frac{1}{20} \rightarrow \boxed{\text{C}}$. ☐

8. A function f is defined over the set of all positive integers and satisfies

$$f(1) = 2022$$

   and

$$f(1) + f(2) + ... + f(n) = n^2 f(n) \text{ for all } n > 1.$$

   Calculate the exact value of $f(2022)$.

   Proof. Answer $f(2022) = 2/2023$
   $f(1) + f(2) + ... + f(n) = n^2 f(n)$
   $f(1) + f(2) + ... + f(n-1) = (n-1)^2 f(n-1)$
   Subtracting these two equations yields $f(n) = n^2 f(n) - (n-1)^2 f(n-1)$ i.e. $(n^2-1)f(n) = (n-1)^2 f(n-1)$, so $f(n) = \frac{n-1}{n+1} f(n-1) \implies f(n) = \frac{4044}{n(n+1)}$. ☐

9. The sum of an infinite geometric series is a positive number S, and the second term in the series is 1. What is the smallest possible value of S?

   (A) $\frac{1+\sqrt{5}}{2}$ **(B)** 2 **(C)** $\sqrt{5}$ **(D)** 3 **(E)** 4

Proof. We have the equations:

$$ar = 1$$
$$\frac{a}{1-r} = S.$$

Thus we want to minimize $S = \frac{\frac{1}{r}}{1-r} = \frac{1}{r-r^2} = k$.

This allows us to write a quadratic in r, specifically $kr - kr^2 = 1 \Rightarrow kr^2 - kr + 1 = 0$. Since r must be real, we can use the discriminant to get $k^2 - 4k \geq 0 \Rightarrow k \geq 4$ when k is positive. Note that $k \neq 0$ since S is positive. We can quickly check that the conditions are satisfied, for an answer of 4. □

10. If the sequence $(a_i)_{i\geq 1}$ is an arithmetic progression, then evaluate

$$\frac{1}{n-1}\sum_{k=1}^{n-1}\frac{1}{\sqrt{a_k}+\sqrt{a_{k+1}}}.$$

[Beginner]

Proof. Answer: $\frac{1}{\sqrt{a_1}+\sqrt{a_n}}$.

Since $\{a_k\}_{k\geq 1}$ is an arithmetic sequence there exists $d \in \mathbb{R}$ such that $a_{k+1} - a_k = d$, therefore

$$\sum_{k=1}^{n-1}\frac{1}{\sqrt{a_k}+\sqrt{a_{k+1}}} = \sum_{k=1}^{n-1}\frac{\sqrt{a_{k+1}}-\sqrt{a_k}}{d} = \frac{1}{d}\left(\sqrt{a_n}-\sqrt{a_1}\right) = \frac{n-1}{\sqrt{a_1}-\sqrt{a_n}}$$

□

11. The numbers $1!, 2!, 3! \ldots 100!$ are written on a blackboard. Is it possible to remove a number so that the product of the remaining numbers is a perfect square? (Formatting suggestion: type the number which is being removed. If it is impossible, type -1).

Proof. Answer: 50! The product of all the factorials contain each odd number an even number of times, and each even number an odd number of times. Therefore we have the product

$$\begin{aligned} X &= N^2 \times (2 \times 4 \times 6 \times \cdots \times 100) \\ &= N^2 \times 2^{50} \times 50! \\ &= (2^{25}N)^2 \times 50! \end{aligned}$$

Therefore removing 50! gives us a square. □

12. Find all integers k such that k+1 and 16k+1 are perfect squares.

Proof. Answer: $k = 3, k = 0$.
Let $k+1 = m^2, 16k+1 = n^2$ for integers m and n. Then:
$16k+16 = 16m^2, 16k+1 = n^2$. Subtracting,
$15 = (4m+n)(4m-n)$, after which, by analysing factors of 15, that the pair $(m^2, n^2) = (4, 49), (1, 1)$. This corresponds to values $k = 3, k = 0$ $\square$

13. Let x, y, z be real numbers such that

$$\begin{aligned} x+y+z &= 1, \\ x^2+y^2+z^2 &= 2, \\ x^3+y^3+z^3 &= 3. \end{aligned}$$

Then find the value of $x^5+y^5+z^5$.

Proof. Answer: 6. We see that $a_n = x^n+y^n+z^n$ can be expressed as a linear homogeneous recurrence of order 3 $a_{n+3} = a_{n+2} + a_{n+1}/2 + a_n/6$ for natural $n \geq 0$. $\square$

14. Let α and β be positive integers such that

$$\frac{43}{197} < \frac{\alpha}{\beta} < \frac{17}{77}.$$

Find the minimum possible value of β.

(Intermediate)

Proof. Answer $\beta = 32$ Note that

$$\frac{77}{17} = 4 + \frac{9}{17} < \frac{\beta}{\alpha} < \frac{197}{43} = 4 + \frac{25}{43},$$

Thus $4 < \frac{\beta}{\alpha} < 5$. Since α and β are positive integers, we may write $\beta = 4\alpha + x$, where $0 < x < \alpha$. Now we get

$$4 + \frac{9}{17} < 4 + \frac{x}{\alpha} < 4 + \frac{25}{43}.$$

So $\frac{9}{17} < \frac{x}{\alpha} < \frac{25}{43}$; that is, $\frac{43x}{25} < \alpha < \frac{17x}{9}$.
We find the smallest value of x for which α becomes a well-defined integer. For $x = 1, 2, 3$ the bounds of α are respectively

$$\left(1 + \frac{18}{25}, 1 + \frac{8}{9}\right)$$

$$\left(3 + \frac{11}{25}, 3 + \frac{7}{9}\right)$$

$$\left(5+\frac{4}{9}, 5+\frac{2}{3}\right)$$

. None of these pair contain an integer between them.

But for $x = 4$, we have that $\frac{43x}{25} = 7 + \frac{5}{9}$, in which case $\alpha = 7$ and $\beta = 32$. Note that this must indeed be the minimum since for $x \geq 5$ we have $\alpha > 8$ and so $\beta > 37$.

□

15. Let $d(n)$ denote the number of positive integers that divide n, including 1 and n. For example, $d(1) = 1, d(2) = 2$, and $d(12) = 6$. (This function is known as the divisor function.) Let

$$f(n) = \frac{d(n)}{\sqrt[3]{n}}$$

There is a unique positive integer N such that $f(N) > f(n)$ for all positive integers $n \neq N$. Find N.

Proof. Answer: 2520. Let $N = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$. Then

$$f(N) = \frac{(a_1+1)(a_2+1)\dots(a_k+1)}{p_1^{a_1/3} p_2^{a_2/3} \dots p_k^{a_k/3}}$$

We can look at each prime power individually. For $p_k > 7$, $\frac{a_k+1}{p_k^{a_k/3}} < 1$, and so should not be a factor of our maximal N. Finding the maximum value of the expression for powers of $2, 3, 5, 7$ yield $2^3, 3^2, 5^1, 7^1$, whose product is 2520. □

16. Find all real functions, $f : \mathbb{R} \to \mathbb{R}$ which satisfy

$$f(x+y)^2 = f(x)^2 + f(y)^2, \ \forall x, y \in \mathbb{R}.$$

[Beginner]

Proof. Answer: $f(x) = 0$ for all reals. Note since

$$f(x+y)^2 = f(x)^2 + f(y)^2, \ \forall x, y \in \mathbb{R},$$

we could set $x = y = 0$ which gives us that $f(0) = 0$, and then $x = -y$ implies that $f(x)^2 = -f(y)^2$ the RHS of the equation is ≥ 0 while the left one is ≤ 0 therefore the only possible solution is when both are 0. Hence , $f(x) = 0, \forall x \in \mathbb{R}$ is the only real function with the given functional property.

□

17. Let p be an odd prime. Find all pairs of positive integers (m, n) satisfying

$$(p-1)(p^n+1) = 4m(m+1).$$

Proof. Answer: $(\frac{p-1}{2}, 1)$ only.

$$p^{n+1} - p^n + p - 1 = 4m^2 + 4m$$
$$p(p^n - p^{n-1} + 1) = (2m+1)^2$$

Note that since RHS is a square, this implies $p^2 \mid LHS$, so $p \mid p^n - p^{n-1} + 1$. This is only possible when $n - 1 = 0$, so $n = 1$. Substituting back into the equation gives $m = \frac{p-1}{2}$. $\square$

18. (Schur) Every non-constant integer polynomial P has an infinite number of prime divisors.

(Beginner)

Proof. Consider $P \in \mathbb{Z}[x]$, and therefore we have that

$$P(x) = a_n x^n + \cdots + a_0,$$

$a_0 = P(0)$, therefore we have that

$$P(P(0)x) = P(0)(1 + Mx),$$

for some M. Hence if we assume that there are only finite numbers of prime divisors $P = \{p_1, p_2, \cdots, p_n\}$ then considering

$$P(P(0)p_1 \cdots p_n) = P(0)(1 + Mp_1 \cdots p_n)$$

notice the second term must be divisible by a prime that's not in the set P. Hence this contradict the finitude of the set P. $\square$

19. (Beginner) As the ruler of the kingdom of Wuturia, you have invited all the rulers from neighbouring kingdoms to a diplomatic drinking party tomorrow night, in 24 hours. However, it has just come to your attention that there has been an attempt to poison all the rulers, and exactly one of the many B barrels of wine has been poisoned such that anyone who drinks it is bound to die in their next sleep. However, there is nobody who knows which barrel is the one that's been poisoned. You know that having any leader die, or having to postpone the event would undoubtedly spark war between the kingdoms. As such, this event must go ahead, but with the poisoned barrel identified and removed.

You are willing to have your servants test all the B barrels by drinking them tonight, and seeing who lives and who dies to single out which barrel was poisoned. However, since you do not wish to publicise the fact that you are subjecting your servants to such fate, nor do you want to risk this news getting out that your oversight allowed the drinks to be poisoned in the first place, your aim to find the minimal number of servants, n to be able to single out exactly which barrel contains the poison.

Additionally, each barrel is hypothetically big enough that it can be trialled by every servant, and still have enough left for the party. As such, it is not a concern to minimise the total number of trials, nor to minimise the worst case scenario for deaths. Also, all deaths are assumed due to the poison and naps are not allowed in this kingdom.

What is the minimum n for the following B?

i) 10

ii) 1000

iii) B (give the answer to this as an equation in terms of variable B)

Proof. Answer: The trick lies in "binary", since by the morning, each servant will either be alive or dead. As such, you can number all the barrels in base 2 starting from 0, with leading zeroes so that all numbers are the same length as the largest number. Then, each servant can be assigned to one magnitude in the base two numbers, and drink any wines where there assigned digit is a "1", and leave any where it is a "0" untouched. Then, by seeing which servants have lived and which have died, a binary representation can be found for the poisoned barrel, and it can be promptly removed before the party. Therefore, the answer is to find the minimum n such that $2^n \geq B$, such that there is enough "bits" to store all of B.

For example, for the case of $B = 10$, where $2^4 > 10 > 2^3$, we need 4 servants, and a 4 digit base 2 numbering of the barrels. This numbering is [0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, 1001]. Person A will drink only [1000, 1001], person B will drink [0100, 0101, 0110, 0111] etc. Then, if servants B and D die, but A and C are still alive, we can deduce that the poisoned barrel was [0101], since those are the two people who drank it. For every barrel which is poisoned, there is one unique combination of deaths which can point out the poisoned barrel.

Note that this method can have nobody die, everyone die (if there's the right number of total barrels), and also subjects people to different risks. □

20. (Hard) Let $f : \mathbb{R} \to \mathbb{R}$ be such that

$$f\left(2x - \frac{f(x)}{\alpha}\right) = \alpha x, \forall\ x \in \mathbb{R}.$$

Prove that $f(x) = \alpha(x - c)$, for all $x \in \mathbb{R}$.

Note: $\alpha \neq 0$.

Proof. Note that

$$g(g(x)) = 2g(x) - x, \tag{1}$$

implying that g is continuous. Moreover it also shows that $g(x)$ is injective, where $g = 2x - f(x)/\alpha$.

Coupled with continuity we have already proven strict montonicity of g , however g cannot be strictly decreasing since for that to occur we need $x < y \implies g(y) < g(x) \implies g(g(x)) < g(g(y)) \implies 2g(x)-x < 2g(y)-y$ which leads to a contradiction hence we have that g must indeed be strictly increasing .

Another observation is that

$$g^3(x) = g^2(g(x)) = 2g^2(x) - g(x), \forall x \in \mathbb{R}, \tag{2}$$

therefore by induction

$$g^n(x) = ng(x) - (n-1)x,\ \forall x \in \mathbb{R},\ n \geq 1. \tag{3}$$

Thus we get that

$$g^n(x) - g^n(0) = n(g(x) - x - g(0)) + x. \tag{4}$$

Since g is increasing, we see that $g^n(x) - g^n(0) > 0$ if $x > 0$ and $g^n(x) - g^n(0) < 0$ if $x < 0$. Using (5) we see that

$$\frac{g^n(x) - g^n(0)}{n} = g(x) - x - g(0) + \frac{x}{n}, \lim_{n\to\infty} \frac{g^n(x) - g^n(0)}{n} = g(x) - x - g(0),$$

for all x. However for $x > 0$ and $x < 0$

$$\lim_{n\to\infty} \frac{g^n(x) - g^n(0)}{n} \geq 0,$$
$$\lim_{n\to\infty} \frac{g^n(x) - g^n(0)}{n} \leq 0,$$

respectively. These imply that

$$\begin{cases} g(x) \geq g(0) + x, x > 0 \\ g(x) \leq g(0) + x.x < 0 \end{cases}$$

and that leads to g being a surjection.

Let $s = \inf\{g(x) : x \in \mathbb{R}$ and $r = \sup\{g(x) : x \in \mathbb{R}\}$. Suppose that $r < \infty$ (r is finite), then we that $r + 1 - g(0) > 0$,

$$g(r + 1 - g(0)) \geq g(0) + r + 1 - g(0),$$

which contradicts the superiority of r. Hence $r = \infty$. Likewise one can prove that $s = -\infty$. Hence $\mathrm{Im}(g) = (-\infty, \infty)$. So g is onto.

Therefore we see that (4),

$$g^n(x) - g^n(0) = n(g(x) - x - g(0)) + x, n \in \mathbb{Z}, x \in \mathbb{R}.$$

Since g is a bijection letting $n \to -\infty$ we see due to the above that $g(x) \leq g(0) + x, x > 0$ and $g(x) \geq g(0) + x, x < 0$ hence $g(x) = x + g(0)$.

□

Proof. A second proof goes as follows. Define the function $h(x)$ by $h(x) = x - \frac{f(x)}{\alpha}$. Since f is continuous then h is continuous. Since $\frac{f(x)}{\alpha} = x - h(x)$ then from the given condition we obtain $f(2x - (x - h(x)) = \alpha x$, so then $f(x + h(x)) = \alpha x$. Since $f(x) = \alpha(x - h(x))$ then

$$\alpha(x + h(x) - h(x + h(x)) = \alpha x.$$

Hence, $h(x + h(x)) = h(x)$ for all reals x.

Now we prove by induction that $h(x + nh(x)) = h(x)$ for all positive integers n and all reals x. Suppose that for some positive integer k that $h(x + kh(x)) = h(x)$ for all reals x. Then

$$h\Big(x+(k+1)h(x)\Big) = h\Big(x+kh(x)+h(x)\Big) = h\Big((x+kh(x))+h(x+kh(x))\Big) = h(x+kh(x)) = h(x),$$

which completes the induction.

We will prove that $h(x)$ must be a constant function. Suppose the contrary.

□