CPMSOC Mathematics Problem Set 2 Solutions

April 6, 2021

1 Problem Set 2 Solutions

1. Prove or disprove: For all pairs of positive integers (a, b), there exists some positive integer n such that an is a perfect cube, while bn is a perfect fifth power.

   Solution

   Let $n = a^5b^9$. Then, $an = a^6b^9 = (a^2b^3)^3$ and $bn = a^5b^{10} = (ab^2)^5$. Therefore, an is a perfect cube and bn is a perfect fifth power, so such a positive integer n always exists. $\square$

2. Determine all positive integers relatively prime to the terms of the infinite sequence $a_n = 2^n + 3^n + 6^n - 1$, where $n \geq 1$.

   Solution

   We claim that $p|a_{p-2}$ for all primes $p > 3$. Indeed, since $2, 3$ and 5 are coprime to p, then by Fermat's Little theorem we have $2^{p-1} \equiv 3^{p-1} \equiv 6^{p-1} \equiv 1 \pmod{p}$. This then gives us

   $$2^{p-2} + 3^{p-2} + 6^{p-2} - 1 \equiv \frac{1}{2} + \frac{1}{3} + \frac{1}{6} - 1 \equiv 0 \pmod{p}.$$

   Consider any integer $x > 1$ We know that x has some prime divisor q. If $q = 2$ or $q = 3$, we get that x is not relatively prime to $a_2 = 48$. If $q > 3$, then since $q|a_{q-2}$, then x will not be relatively prime to a_{q-2}. Hence, only 1 is relatively prime to all terms in the sequence. $\square$

3. A deck of $n > 1$ cards is given. A positive integer is written on each card. The deck has the property that the arithmetic mean of the numbers on each pair of cards is also the geometric mean of the numbers on some collection of one or more cards. For what n does it follow that the numbers on all the cards are equal?

Solution

Suppose that $a_1 \ldots a_n$ satisfy the required properties but are not all equal. If $d = \gcd(a_1, a_2 \ldots a_n) > 1$ then replace $a_1, a_2 \ldots a_n$ by $\frac{a_1}{d}, \frac{a_2}{d} \ldots \frac{a_n}{d}$. Hence without loss of generality we may assume

$$\gcd(a_1, a_2 \ldots a_n) = 1.$$

Also WLOG we may assume

$$a_1 \geq a_2 \geq \ldots \geq a_n.$$

As $a_1 \geq 2$, let p be a prime divisor of a_1. Let k be the smallest index such that $p \nmid a_k$ (which must exist). In particular note that $a_1 > a_k$.
Consider the mean $x = \frac{a_1+a_k}{2}$. By assumption, it equals some geometric mean, hence

$$\sqrt[m]{a_{i_1} a_{i_2} \ldots a_{i_m}} = \frac{a_1 + a_k}{2} > a_k.$$

Since the arithmetic mean is an integer not divisible by p, all the indices $i_1, i_2 \ldots i_m$ must be at least k. But then the geometric mean is at most a_k, contradiction. $\square$

4. An integer sequence is defined by $a_n = 2a_{n-1} + a_{n-2}, (n > 1), a_0 = 0$, $a_1 = 1$. Prove that 2^k divides a_n if and only if 2^k divides n.

Solution

If we analyse the terms of the sequence modulo 4, it is easy to verify by induction that a_{2n} is even and $a_{2n+1} \equiv 1 \pmod 4$. for all $n \geq 0$ (the terms of the sequence modulo 4 are $0, 1, 2, 1, 0, 1, 2, 1 \ldots$). We can also check by strong induction on t that $a_{n+t} = a_{t+1}a_n + a_t a_{n-1}$ for $t \geq 0$. It is easy to check that the base cases $t = 0, t = 1$ are true.

If the identity is true for for all $t \leq k$ $(k \geq 1)$ then for $t = k + 1$ we have

$$\begin{aligned}
a_{n+k+1} &= 2a_{n+k} + a_{n+k-1} \\
&= 2(a_{k+1}a_n + a_k a_{n-1}) + (a_k a_n + a_{k-1}a_{n-1}) \\
&= (2a_{k+1} + a_k)a_n + (2a_k + a_{k-1})a_{n-1} \\
&= a_{k+2}a_n + a_{k+1}a_{n-1}.
\end{aligned}$$

Thus, the identity is proven by strong induction. Using the identity, we have in particular that

$$a_{2n} = a_n(a_{n+1} + a_{n-1}).$$

From the modulo 4 analysis, note that $2^k | a_n$ iff $2^k | n$ for $k = 0, 1, 2$. Assume (for means of strong induction) that $2^k | a_n$ iff $2^k | n$ for all n for $k = 0, 1, 2 \ldots m$ for some $m \geq 2$. Then suppose that $2^{m+1} | n$. Let $p = \frac{n}{2}$. We have

$$a_{2p} = a_p(a_{p+1} + a_{p-1}).$$

Since $2^m | a_p$ (as $2^m | p$) and since $a_{p+1} + a_{p-1}$ is even (as $p + 1$ and $p - 1$ are odd so a_{p+1} and a_{p-1} are odd), then a_n is divisible by 2^{m+1}.

Now suppose that a_n is divisible by 2^{m+1}. Then a_n is divisible by 2^m so n is divisible by 2^m. Let $p = \frac{n}{2}$. We again have

$$a_{2p} = a_p(a_{p+1} + a_{p-1}).$$

Assume for the sake of contradiction that n is not divisible by 2^{m+1}. Then p is divisible by 2^{m-1} but not 2^m, so a_p is divisible by 2^{m-1} but not by 2^m. Also $p + 1$ and $p - 1$ are odd, so $a_{p+1} + a_{p-1}$ is congruent to 2 modulo 4, so a_{2p} is divisible by 2^m but not by 2^{m+1}, contradiction. Therefore, n must be divisible by 2^{m+1}. This completes the inductive step, so the proof is complete. $\square$

5. Find all pairs of integers (a, b) for which there exist functions $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ satisfying $f(g(x)) = x + a$ and $g(f(x)) = x + b$ for all integers x.

Solution

The answer is if $a = b$ or $a = -b$. In the former case, we can take $f(x) \equiv x + a$ and $g(x) \equiv x$. In the latter case, we can take $f(x) \equiv -x + a$ and $g(x) \equiv -x$.

Now we prove that these are the only possibilities. We first see that the functions f and g are bijections. Surjectivity is immediate from definition of f and g. To prove injectivity, notice that if $f(u) = f(v)$ then

$$g(f(u)) = g(f(v)) \implies u + b = v + b \implies u = v, \text{and similarly for } g.$$

Note that for any x, we have

$$f(x + b) = f(g(f(x)) = f(x) + a.$$

$$g(x + a) = g(f(g(x)) = g(x) + b.$$

If either a or b is zero, we immediately get the other is zero and hence done. So assume $ab \neq 0$.

If $|b| > |a|$ then two of

$$\{f(0), f(1), \ldots f(b-1)\} \pmod{a}$$

coincide, which together with repeatedly applying the first equation above will then give a contradiction to injectivity of f. A similar contradiction is reached if $|a| > |b|$ by symmetry. This completes the proof. $\square$

6. Find all pairs of positive integers $m, n \geq 3$ for which there exist infinitely many positive integers a such that

$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$

is itself an integer.

Solution.

The condition is equivalent to $a^n + a^2 - 1$ dividing $a^m + a - 1$ as polynomials.

Claim: We must have $m \leq 2n$.

Proof. Assume the contrary $m > 2n$ and let $0 < r < 1$ be the unique real number with $r^n + r^2 = 1$, hence $r^m + r = 1$. But now

$$\begin{aligned} 0 &= r^m + r - 1 < r(r^n)^2 + r - 1 = r\left((1 - r^2)^2 + 1\right) - 1 \\ &= -(1 - r)\left(r^4 + r^3 - r^2 - r + 1\right). \end{aligned}$$

As $1 - r > 0$ and $r^4 + r^3 - r^2 - r + 1 > 0$, this is a contradiction.

Clearly $m > n$.

$$\begin{aligned} & a^n + a^2 - 1 \mid a^m + a - 1 \\ \iff & a^n + a^2 - 1 \mid (a^m + a - 1)(a + 1) = a^m(a + 1) + (a^2 - 1) \\ \iff & a^n + a^2 - 1 \mid a^m(a + 1) - a^n \\ \iff & a^n + a^2 - 1 \mid a^{m-n}(a + 1) - 1. \end{aligned}$$

The right-hand side has degree $m - n + 1 \leq n + 1$, and the leading coefficients are both $+1$. So the only possible situations are

$$\begin{aligned} a^{m-n}(a + 1) - 1 &= (a + 1)\left(a^n + a^2 - 1\right) \\ a^{m-n}(a + 1) + 1 &= a^n + a^2 - 1. \end{aligned}$$

The former fails by just taking $a = -1$; the latter implies $(m, n) = (5, 3)$. As our work was reversible, this also implies $(m, n) = (5, 3)$ works, so the proof is complete. $\square$