Competitive Programming and Mathematics Society

Introduction to Competitive Mathematics

Workshop 1, Week 4, Term 1, 2021

CPMSoc Mathematics

Overview

CPMSOC

What is Competitive Mathematics?

Competitive Mathematics at UNSW

CPMSOC

- Simon Marais Mathematics Competition (SMMC)
  - Happens in mid-October
  - Sign up by August through the university
  - Mathematics department posts information for signing up closer to the date
  - More than $50,000 in individual prizes, and internship offers.

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- Meet people
- Having fun
- Being employable
- Solving real-world problems

"Mathematics is distilled thinking" - Poh Shen-Loh

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The Problems

- Proof-based
- Solutions are much easier to explain than to invent
- Require unusual techniques, observations, or combinations
- Usually chosen to be aesthetically pleasing

Subject areas

CPMSOC

- Number theory
- Geometry
- Polynomials
- Vectors
- Complex numbers
- Calculus
- Functional equations
- Inequalities
- Combinatorics & probability
- Combinatorial games
- Analysis
- Trigonometry
- Graph theory
- Group theory

Problems vs. Exercises

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"An **exercise** is a question that tests the student's mastery of a narrowly focused technique ... the **path towards solution is always apparent**. In contrast, a **problem** is a question that cannot be answered immediately. Problems are often open-ended, paradoxical, and sometimes unsolvable, and **require investigation before one can come close to a solution**. Problems and problem solving are at the heart of mathematics."

– Paul Zeitz

Learning Competitive Mathematics

How to Learn Problem-solving

- Practice strategically
- Read solutions

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Strategies

- Select good notation
- Modify the problem
- Work backwards
- Consider concrete examples
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Techniques

- Induction
- Invariants
- Cases/exhaustion
- Pigeonhole principle
- Extremal principle
- Telescoping
- Symmetry
- Generating functions

Resources

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Books

- Problem Solving Tactics (Pasquale et. al., 2014)
- The Art and Craft of Problem Solving (Zeitz, 2007)
- Putnam and Beyond (Gelca & Andreescu, 2007)
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Other

- YouTube
- The Putnam Archive
- Wikipedia
- Discord (discord.gg/rPFkGg3PT6)

Proof by Contradiction

General Form

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Proof by Contradiction

1. Suppose P.
2. Q follows from P. $(P \implies Q)$
3. Q is false. $\neg Q$
4. P is false. $\neg P$

Theorem (Law of Contraposition)

$$P \to Q \iff \neg Q \to \neg P$$

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  Note also that q is not divisible by any prime, since division by p_i leaves remainder 1.
- Any composite number is divisible by a prime, so q is divisible by a prime.
- Therefore, q does not exist. So, there must be infinitely many primes.

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Let $a_1, a_2, \ldots, a_n$ be a rearrangement of the numbers $1, 2, \ldots, n$.
Show that if n is odd then $A = (a_1 - 1)(a_2 - 2) \cdots (a_n - n)$ is even.

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- $1, 2, 3, \ldots, n$ contains $(n-1)/2$ even numbers and $(n+1)/2$ odd numbers.
- Thus, A is even.

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- This is a contradiction, so the overlap of some two surfaces must be at least $1/9$.

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- Thus, we cannot partition $[0, 1]$ in this way.

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- So, $\sqrt{30}={}^{p}/_{q}$ where $\gcd(p,q)=1$ and $q\neq 0$.

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- So, $\sqrt{30}={}^{p}/_{q}$ where $\gcd(p,q)=1$ and $q\neq 0$. Squaring and rearranging gives $p^2=30q^2$.
- Then p^2 is even, and so p is too. But then p^2 has a factor of 4.

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- Squaring this gives $6+5r^2+2r\sqrt{30}$, which must be rational. Again, we see that $\sqrt{30}$ must be rational.
- So, $\sqrt{30}={}^{p}/_{q}$ where $\gcd(p,q)=1$ and $q\neq 0$. Squaring and rearranging gives $p^2=30q^2$.
- Then p^2 is even, and so p is too. But then p^2 has a factor of 4.
- Since $q^2={}^{p^2}/_{30}$, q^2 has a factor of 2, so q is even.

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- Thus, $\gcd(p,q)\neq 1$, deriving a contradiction.
- That is, $\sqrt{2}+\sqrt{3}+\sqrt{5}$ is irrational.

Example

We call a 5-tuple of integers arrangeable if its elements can be labelled a, b, c, d, e in some order so that $a - b + c - d + e = 29$. Determine all 2017-tuples of integers $n_1, n_2, \ldots, n_{2017}$ such that if we place them in a circle in clockwise order, then any 5-tuple of numbers in consecutive positions on the circle is arrangeable.

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- First note that we can subtract 29 from each n_i so that the condition becomes $a-b+c-d+e=0$ instead.
- Now if we add five consecutive n_i, we get

$$\begin{aligned} n_i + n_{i+1} + n_{i+2} + n_{i+3} + n_{i+4} &= (n_i - n_{i+1} + n_{i+2} - n_{i+3} + n_{i+4}) + 2(n_{i+1} + n_{i+3}) \\ &= 2(n_{i+1} + n_{i+3}), \end{aligned}$$

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- Replacing i with $i + 1$ and subtracting, we find that $n_i - n_{i+5}$ is also even.

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- Since $n_i - n_{i+5}$ is even, n_i and n_{i+5} have the same parity.

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- Since $n_i - n_{i+5}$ is even, n_i and n_{i+5} have the same parity.
- Then $n_1, n_6, \ldots, n_{2016}$ have the same parity. But the numbers are arranged clockwise, so n_{2016} and n_4 have the same parity, and the pattern continues, until we find that every n_i has the same parity.

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- Now, since $n_i + n_{i+1} + n_{i+2} + n_{i+3} + n_{i+4}$ is even, we can conclude that every n_i is even.
- Now we can suppose that we have some solution for which $\sum_{i=1}^{2017} |n_i| = S$ is minimal.

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- If $S > 0$, then this solution has a smaller absolute sum of $S/2$, so must contradict the minimality of the original solution.

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- Since every n_i is even, we can take ${n_i}/2$ as a new solution.
- If $S > 0$, then this solution has a smaller absolute sum of ${S}/2$, so must contradict the minimality of the original solution.
- Thus, $S = 0$, but then $n_i = 0$.
- We subtracted 29 from each n_i originally, so the only solution is then $n_i = 29$ for all i.